

Dynamical Properties of Weierstrass Elliptic Functions on Square Lattices

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2010

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ABSTRACT

JOSHUA J. CLEMONS: Dynamical Properties of Weierstrass Elliptic Functions on
Square Lattices

(Under the direction of Prof. Jane Hawkins)

Abstract. In this dissertation we prove that the Julia set of a Weierstrass elliptic function on a square lattice is connected. We further show that the parameter space contains an infinite number of Mandelbrot sets. As a consequence, this proves the existence of Siegel disks and gives a description of the bifurcation locus about super-attracting parameters corresponding to super-attracting fixed points. We conclude with a description of a family of rational maps that approximate the Weierstrass elliptic function on a square lattice.

ACKNOWLEDGEMENTS

I would like to briefly thank all those that have supported me through this long process. In particular, I would like to thank my advisor, Jane Hawkins, for her direction and inspiration. I am extremely grateful for all the long hours that were put into reading and re-reading my work. Additionally, I would like to thank my committee for their helpful suggestions and their willingness to talk with me. Finally, my wife of almost seven years has been with me through it all. Her support has been invaluable. Thank you.

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CHAPTER 0

Introduction

It is our goal in this thesis to present a detailed description of the dynamics of Weierstrass elliptic functions on square lattices. To do so, we show that the dynamics are intimately related to those of the classical quadratic map, $z^2 + c$. When studying and classifying holomorphic families of maps, a natural starting point is to look at degree 1 rational maps. These form the class of conformal biholomorphisms of the sphere. This case is relatively elementary and can be worked out explicitly. The next step is to look at rational maps of degree 2. The simplest of these is the family of quadratic polynomials. We show that Weierstrass elliptic functions on square lattices share many dynamical properties with the family of quadratic polynomials.

When studying the dynamics of elliptic functions, one would naturally ask: “What are the simplest elliptic functions?” We argue that the Weierstrass elliptic functions are the simplest to study for the following reason. The elliptic functions on a fixed lattice form a finitely generated field. That field can be generated by the Weierstrass elliptic function and its derivative [26]. Taking the perspective that every elliptic function is built out of the Weierstrass elliptic function and its derivative, it follows that one would like to understand the dynamics of Weierstrass elliptic functions to better understand the dynamics of elliptic functions in general.

In complex dynamics, the classification and understanding of the critical orbits can to some degree classify the dynamics. For rational maps this is described in [4, 41] and for more general meromorphic functions in [5]. A basic example is that the basin of attraction always contains a critical value [4, 5]. The unique feature of Weierstrass elliptic functions on square lattices is that there is only one “free” critical orbit, even though it has three critical values. One of the critical values is 0, a pole, and the other two collapse after one iterate to the same value. The critical value of 0 is mapped to ∞ and can no longer be iterated, leaving the other critical orbit to determine the dynamics. This implies, for instance, that the function can have at most one non-repelling cycle [5]. Typically a Weierstrass elliptic function can have as many as three distinct critical orbits and so as many as three non-repelling cycles. (Of note is the triangular case, which we do not study here. In this case one can say that these functions also have “one critical orbit” since the critical orbits are $2\pi/3$ rotations of each other [33, 32, 35].)

We begin with introducing Weierstrass elliptic functions and discussing their mapping properties. As mentioned previously we have a sharp interest in the critical points and critical values. We then show in Chapter 2 that the Julia set of a Weierstrass elliptic function on a square lattice is connected (Theorem 2.2.1). This was known in the case of a real super-attracting fixed point [31] or a cycle of Siegel disks [33]. If all cycles are repelling, then the Fatou set is empty and the Julia set is the whole sphere. This can occur and is dynamically interesting. We refer the reader to [31] and especially [30] (and the references therein). As a consequence of connecting the dynamics of Weierstrass elliptic functions on square lattices to the classical quadratic family we demonstrate the existence of Siegel disks (discussed in Chapter 3). It was known that these occur for elliptic functions of the form $\wp_\Gamma(z) + b$ where Γ is triangular [32].

The Mandelbrot set is defined to be the set $\{c : p_c^n(0) \text{ is bounded}\}$ where $p_c(z) = z^2 + c$. To identify an analog to this set in the parameter space of Weierstrass elliptic functions on square lattices we let λ be a non-zero complex number and define \wp_λ to be the Weierstrass elliptic function defined on the square period lattice determined by λ and λi . We then consider the following set:

$$\{\lambda : \text{the critical orbit of } \wp_\lambda \text{ is "contained"}\}$$

and show that it contains infinitely many copies of the Mandelbrot set. We define the notion of “contained” in Chapter 3.

In Chapter 4 we prove preliminary symmetry results and conjugacy results between singular perturbations and two-term Laurent expansions about the origin of Weierstrass elliptic functions. Singular perturbations are studied extensively by Devaney et al. in [6, 17, 19] and the references therein. We prove a theorem similar to Theorem 2.2.1 and discuss the structure of the parameter spaces of the approximations as they relate to the parameter space of Weierstrass elliptic functions.

The main results of this thesis are as follows. In Chapter 2 we prove that given any square lattice Ω the Julia set of the Weierstrass elliptic function, \wp_Ω , is connected. This is Theorem 2.2.1. In Chapter 3 we prove three main results which build on each other. We first prove the Weierstrass elliptic functions on square lattices are quadratic-like (Theorem 3.2.8). After showing that the quadratic-like mappings are analytic families (Theorem 3.2.11) we show that the parameter space contains Mandelbrot sets (Theorem 3.2.19).

CHAPTER 1

Weierstrass Elliptic \wp Functions

The dynamics of the family of Weierstrass elliptic \wp functions the main object of study in this thesis. In order to understand the dynamics of \wp for a given lattice we must first analyze how it maps. However, because the function is not algebraic, it is more challenging to evaluate explicit values of the function. In contrast, the Weierstrass elliptic functions do have strong algebraic properties that allow us to analyze their values more thoroughly than one would expect.

While the rational functions form the class of meromorphic functions defined on the Riemann sphere, the elliptic functions form the class of meromorphic functions defined on complex tori, called elliptic curves. We approach these functions as objects of dynamical interest instead of just meromorphic functions on elliptic curves. One familiar with elliptic function theory needs to keep in mind that two lattices that correspond to conformally homeomorphic elliptic curves may produce Weierstrass elliptic \wp functions with very different dynamical properties: the dynamical properties of elliptic functions depend on more than the equivalence class of the analytic structure on an elliptic curve.

1.1. Lattices

The Weierstrass \wp function is considered to be the most basic of the family of elliptic functions. With that consideration we would like to study it in detail so that we may

better understand other elliptic functions. Let \mathbb{C} denote the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere.

DEFINITION 1.1.1. *A lattice Ω is an additive subgroup of \mathbb{C} with two generators that is isomorphic to \mathbb{Z}^2 , such that the generators $\omega_1, \omega_2 \in \mathbb{C}^*$ are \mathbb{R} linearly independent:*

$$(1.1) \quad \Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

We denote this $\Omega = \langle \omega_1, \omega_2 \rangle$.

The *residue class* of a point z with respect to a lattice Ω is the set of values

$$(1.2) \quad z + \Omega = \{z + \omega : \omega \in \Omega\},$$

and a *fundamental region* is a simply connected region in \mathbb{C} that contains a representative of each residue class of each point. The most commonly used example of a fundamental region is a *period parallelogram*, which is defined as:

$$P_\Omega = \{t\omega_1 + s\omega_2 : 0 \leq t, s < 1\}.$$

The generators of a lattice are by no means unique.

PROPOSITION 1.1.2. ([26]) *Two lattices Ω and Ω' are equal if and only if*

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

REMARK 1. *Given a lattice Ω , one must choose generators ω_1 and ω_2 before a period parallelogram can be defined.*

We say that two lattices are *similar* if $\Omega' = \lambda\Omega$ for some $\lambda \in \mathbb{C}^*$. The similarity classes form equivalence classes of lattices, and we primarily study parameterized dynamics within a similarity class. The natural question that arises is the issue of how to index the different similarity classes uniquely. We begin by noting that the lattice $\langle \omega_1, \omega_2 \rangle$ is similar to $\langle 1, \omega_2/\omega_1 \rangle$. Thus, one can always consider the lattice $\langle 1, \tau \rangle$ as a representative of a similarity class, i.e. we can parameterize similarity classes by $\tau \in \mathbb{C}^*$. However, this representative is not unique. Consider the lattices $\langle 1, i \rangle$ and $\langle 1, -i \rangle$, for example. We can assume that $\Im \tau > 0$ since we could replace τ with $-\tau$. Considering the orbit of the similarity classes of $\langle 1, \tau \rangle$ under the $SL_2(\mathbb{Z})/\{I, -I\}$ action in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a+b\tau}{c+d\tau} \\ 1 \end{pmatrix},$$

we have a group action on the upper half plane $z \mapsto \frac{az+b}{cz+d}$. Just as in the case where we choose a fundamental region for a lattice's group action on the plane, a period parallelogram, we can do so again here.

THEOREM 1.1.3. [26] *Every lattice is similar to a lattice of the form $\langle 1, \tau \rangle$ for some unique $\tau \in T$, where*

$$(1.3) \quad T = \{z : \Im(z) > 0, -1/2 \leq \Re(z) < 1/2, |z| \geq 1, (|z| > 1 \text{ when } \Re(z) > 0)\}.$$

The region T in the τ plane described in Theorem 1.1.3 is pictured in Figure 1.1. To specific values of τ we attach names of similarity classes.

DEFINITION 1.1.4. *The following lattice shapes are defined as follows:*

- (1) *A lattice Ω is square if $\tau(\Omega) = i$.*

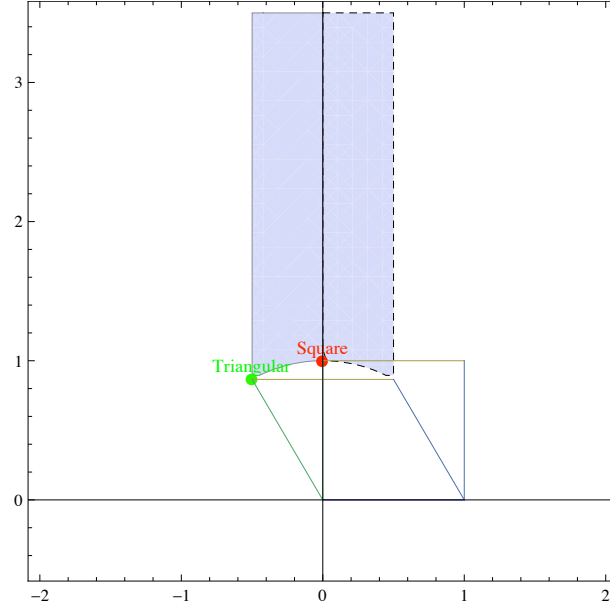


FIGURE 1.1. $\tau \in T$

- (2) A lattice Ω is triangular if $\tau(\Omega) = e^{2\pi i/3}$.
- (3) A lattice Ω is rectangular if $\tau(\Omega)$ is pure imaginary.
- (4) A lattice Ω is rhombic if $|\tau(\Omega)| = 1$

1.2. The Elliptic Functions of Weierstrass

We begin by defining analytic and meromorphic functions.

DEFINITION 1.2.1. Let U be an open subset of \mathbb{C} . We say that $f : U \rightarrow \mathbb{C}$ is analytic if f is complex differentiable at every point of U .

DEFINITION 1.2.2. Let U be an open subset of \mathbb{C} . We say that $f : U \rightarrow \mathbb{C}_\infty$ is meromorphic on U if f is analytic on U except at a set $\{p_i\}$ of isolated points in U , and at these isolated points $\lim_{z \rightarrow p_i} |f(z)| = \infty$.

We say that a meromorphic function is *elliptic* if it is periodic with respect to a lattice, i.e. it is well defined on residue classes.

DEFINITION 1.2.3. *A meromorphic function, $f : \mathbb{C} \rightarrow \mathbb{C}_\infty$, is elliptic if there is a lattice Ω so that $f(z + \omega) = f(z)$ for all $\omega \in \Omega$.*

Liouville's Theorem [2] gives that the only entire (analytic on \mathbb{C}) elliptic functions are the constant functions. For if we suppose that an elliptic function is entire, then it must be bounded by its maximum value on the closure of a fundamental region. Thus a non-constant elliptic function must have poles.

THEOREM 1.2.4. [26] *An elliptic function assumes any value $c \in \mathbb{C}$ on $n \in \mathbb{N}$ residue classes, where n is independent of c and allows for multiplicities at critical values.*

This number n is the *order* of the elliptic function.

DEFINITION 1.2.5. *The order of an elliptic function f on a lattice Ω is $\#\{z \in \mathbb{C}/\Omega : f(z) = c\}$, where c is not a critical value.*

THEOREM 1.2.6. [26] *Every elliptic function has order at least two.*

The Weierstrass elliptic \wp function is an elliptic function of order 2 defined as:

$$(1.4) \quad \wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega^*} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

where $\Omega^* = \Omega \setminus \{0\}$.

The first property to check is periodicity. We check this for $\wp_\Omega(z + \omega_1) = \wp_\Omega(z)$.

Checking $\wp_\Omega(z + \omega_2) = \wp_\Omega(z)$ is identical. Plugging in to the definition, we have

$$\wp_\Omega(z + \omega_1) = \frac{1}{(z + \omega_1)^2} + \sum_{\omega \in \Omega^*} \left(\frac{1}{(z + \omega_1 - \omega)^2} - \frac{1}{\omega^2} \right)$$

and interchanging the first term with the term $\frac{1}{(z+\omega_1-\omega_1)^2}$ we have the result.

One also needs to verify that the series converges for all $z \in \mathbb{C} \setminus \Omega$. This is done in [26].

Let $\langle \omega_1, \omega_2 \rangle$ be denoted by Ω . When scaling the lattice we have the following relations, which we refer to as the *homogeneity equations*. For $k \in \mathbb{C}^*$ we have

$$(1.5) \quad \wp_{k\Omega}(kz) = \frac{1}{k^2} \wp_{\Omega}(z),$$

or similarly,

$$(1.6) \quad \wp_{k\Omega}(z) = \frac{1}{k^2} \wp_{\Omega}\left(\frac{z}{k}\right).$$

These relations follow directly from the definition of \wp_{Ω} and give a very useful tool for analyzing the families $\wp_{\lambda\Omega}$ where $\lambda \in \mathbb{C}^*$. Recall that

$$P_{\Omega} = \{t\omega_1 + s\omega_2 : 0 \leq t, s < 1\}$$

and that $P_{\Omega} + u$ is a fundamental region for any $u \in \mathbb{C}$.

THEOREM 1.2.7. [26] *For any $u \in \mathbb{C}$ the mapping*

$$\wp_{\Omega} : P_{\Omega} + u \rightarrow \mathbb{C}_{\infty}$$

is a 2-1 and onto map except at points in the Ω -orbit of $\{0, \omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2\}$.

Elliptic functions can be thought of as n -fold branched coverings of the Riemann sphere by the torus \mathbb{C}/Ω . In this case, the Weierstrass elliptic \wp functions are 2-1 branched coverings.

THEOREM 1.2.8. ([26]) *The map*

$$(1.7) \quad \wp_{\Omega} : \mathbb{C}/\Omega \rightarrow \mathbb{C}_{\infty}$$

is a 2-1 branched covering.

By differentiating the series defining \wp_Ω we can obtain a series for \wp'_Ω :

$$(1.8) \quad \wp'_\Omega(z) = -2 \sum_{\omega \in \Omega^*} \frac{1}{(z - \omega)^3}.$$

The derivative of \wp_Ω squared can be written as a polynomial of \wp_Ω [26]:

$$(1.9) \quad (\wp'_\Omega(z))^2 = 4\wp_\Omega(z)^3 - g_2(\Omega)\wp_\Omega(z) - g_3(\Omega),$$

where g_2 and g_3 are invariants of the lattice. The invariants g_2 and g_3 can be written as follows:

$$(1.10) \quad g_2(\Omega) = 60 \sum_{\omega \in \Omega^*} \frac{1}{\omega^4}; \quad g_3(\Omega) = 140 \sum_{\omega \in \Omega^*} \frac{1}{\omega^6}$$

and so these invariants also have homogeneity properties:

$$(1.11) \quad g_2(k\Omega) = \frac{1}{k^4}g_2(\Omega); \quad g_3(k\Omega) = \frac{1}{k^6}g_3(\Omega).$$

These invariants also have the property that given complex numbers g_2 and g_3 with $g_2^3 - 27g_3^2 \neq 0$ there exists a unique lattice Ω so that $g_2 = g_2(\Omega)$ and $g_3 = g_3(\Omega)$ [26].

Critical values of \wp_Ω correspond to zeros of $4\wp_\Omega(z)^3 - g_2(\Omega)\wp_\Omega(z) - g_3(\Omega)$. Using the Fundamental Theorem of Algebra we can write

$$(1.12) \quad (\wp'_\Omega(z))^2 = 4(\wp_\Omega(z) - e_1)(\wp_\Omega(z) - e_2)(\wp_\Omega(z) - e_3).$$

It is known that the roots $\{e_1, e_2, e_3\}$ are distinct [26] and are $e_1 = \wp_\Omega(\omega_1/2)$, $e_2 = \wp_\Omega(\omega_2/2)$, and $e_3 = \wp_\Omega((\omega_1 + \omega_2)/2)$.

REMARK 2. *As with the definition of a period parallelogram the labeling of the critical values $\{e_1, e_2, e_3\}$ depends on the choice of generators ω_1 and ω_2 . We adhere to this convention when it is understood that $\Omega = \langle \omega_1, \omega_2 \rangle$.*

The critical values satisfy important algebraic relations obtained by equating like terms in the factorization and the original expression $(\wp'_\Omega(z))^2 = 4\wp_\Omega(z)^3 - g_2(\Omega)\wp_\Omega(z) - g_3(\Omega)$. We list the ones we use most:

$$(1.13) \quad e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}.$$

Because of our focus on square lattices we develop in detail the formulas for the critical values in terms of the lattice.

PROPOSITION 1.2.9. [26] *Suppose that Ω is a square lattice. Then $e_3 = 0$ and $e_2 = -e_1$. Furthermore, $g_2 = 4e_1^2$ and $g_3 = 0$.*

Certain lattices have symmetry about the real axis. These lattices are called *real lattices*.

PROPOSITION 1.2.10. *If Ω is a lattice so that $\bar{\Omega} = \Omega$, where $\bar{\Omega} = \{\bar{\omega} : \omega \in \Omega\}$, then*

$$\wp_\Omega : \mathbb{R} \setminus \Omega \rightarrow \mathbb{R}$$

and

$$\wp'_\Omega : \mathbb{R} \setminus \Omega \rightarrow \mathbb{R}.$$

PROOF. It is sufficient to show that $\overline{\wp_\Omega(z)} = \wp_\Omega(z)$ for all z such that $\bar{z} = z$. From the definition of \wp_Ω we have that

$$\overline{\wp_\Omega(z)} = \wp_{\bar{\Omega}}(\bar{z}),$$

and by assumption $\overline{\Omega} = \Omega$ and $\bar{z} = z$. Thus,

$$\wp_{\overline{\Omega}}(\bar{z}) = \wp_{\Omega}(z).$$

The same argument applies to \wp'_{Ω} . □

PROPOSITION 1.2.11. *Given a lattice Ω , \wp_{Ω} is even.*

PROOF. We show that $\wp_{\Omega}(-z) = \wp_{\Omega}(z)$. Since Ω is an additive group we have $-\Omega = \Omega$. Using the homogeneity equations (1.5) we have

$$\wp_{-\Omega}(-z) = \frac{1}{(-1)^2} \wp_{\Omega}(z) = \wp_{\Omega}(z).$$

□

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$. Putting together Propositions 1.2.11 and 1.2.10 we have the following:

PROPOSITION 1.2.12. *If $\overline{\Omega} = \Omega$, then*

$$\wp_{\Omega} : \mathbb{R} \setminus \Omega \rightarrow \overline{\mathbb{R}^+}.$$

An important benchmark quantity is defined as the following:

DEFINITION 1.2.13. *We define $\gamma > 0$ as*

$$(1.14) \quad \gamma^2 = \wp_{\langle 1, i \rangle}(1/2).$$

This constant is referred to the *lemniscate constant*;

$$\gamma \approx 2.62206\dots,$$

and the exact formula for γ is the following

$$\gamma = \frac{1}{4} \sqrt{\frac{2}{\pi}} (\Gamma(1/4))^2,$$

where $\Gamma(z)$ is the classical Gamma function. Both the decimal approximation and the exact formula can be found in [1].

Using the homogeneity equations and lemniscate constant we can describe the critical values of \wp_Ω when Ω is square. Let $\Omega = \langle \lambda, \lambda i \rangle$ for some $\lambda \in \mathbb{C}^*$. Using (1.5) and the lemniscate constant, γ , we have

$$(1.15) \quad e_1 = \wp_\Omega \left(\frac{\lambda}{2} \right) = \frac{1}{\lambda^2} \wp_{\langle 1, i \rangle}(1/2) = \frac{\gamma^2}{\lambda^2}.$$

This gives the following.

PROPOSITION 1.2.14. *Let $\lambda \in \mathbb{C}^*$. The critical values of $\wp_{\langle \lambda, \lambda i \rangle}$ are*

$$(1.16) \quad \{-e_1, 0, e_1\} = \left\{ -\frac{\gamma^2}{\lambda^2}, 0, \frac{\gamma^2}{\lambda^2} \right\}.$$

1.3. General Elliptic Functions

Given a fixed lattice, Ω the elliptic functions on that lattice form a field. This field is finitely generated and, in fact, is the field generated by \wp_Ω and \wp'_Ω [26]. In addition to noting that \wp has the lowest possible order, this partially justifies the statement that the Weierstrass elliptic \wp functions are the simplest elliptic functions.

THEOREM 1.3.1. [26] *Let f be an elliptic function on a lattice Ω . Then there exist rational functions R and S so that $f(z) = R(\wp_\Omega(z)) + S(\wp_\Omega(z))\wp'_\Omega(z)$.*

From this we can conclude that every elliptic function has a finite number of critical values.

THEOREM 1.3.2. *If f is elliptic then it has finitely many critical values.*

PROOF. From Theorem 1.3.1 and Corollary 1.2.4 we have that $f'(z)$ has finitely many zeros in a fundamental period and thus f has finitely many critical values. \square

1.4. Dynamics of Meromorphic Maps

Much of the theory of meromorphic maps is a generalization of the work on rational maps. However, there are also many results for rational maps that do not hold in this setting. Whereas a rational function is defined and analytic at infinity, a transcendental meromorphic function cannot be defined to be continuous at infinity, much less analytic. Thus transcendental meromorphic functions cannot be iterated on the entire Riemann sphere. This fact alone makes the theory somewhat different. However, we can iterate on the complex plane with the convention that if a point iterates to a pole then its orbit terminates at infinity.

Although it is true that the composition of rational functions is rational, it is not the case that the composition of transcendental meromorphic functions is even meromorphic. This is the case in our setting; the composition of two elliptic functions need not be meromorphic, much less elliptic. One can see this by noting that $\wp_\Omega \circ \wp_\Omega$ has infinitely many poles in a fundamental region of \wp_Ω ; namely each point of $\wp_\Omega^{-1}(\Omega)$ is a pole. This is because each value $\omega \in \Omega$ is attained two times by \wp_Ω on a fundamental region, and so $\wp_\Omega \circ \wp_\Omega$ has infinitely many poles in a fundamental region, which is bounded. The poles of $\wp_\Omega \circ \wp_\Omega$ are therefore not isolated, and so $\wp_\Omega \circ \wp_\Omega$ cannot be meromorphic on \mathbb{C} .

1.5. The Fatou and Julia Sets

The notation for composition in this paper is $f^n(z)$ and does not mean $(f(z))^n$. We always assume that f is not constant or a linear transformation to avoid degenerate cases. Since one cannot define a transcendental function at ∞ to be continuous much less analytic, we need to consider poles and points that arrive at poles in their forward orbits.

DEFINITION 1.5.1. *Define*

$$A_1(f) = \{z : f(z) = \infty\},$$

which we call prepoles of order 1 or just poles;

$$A_n(f) = \{z : f^n(z) = \infty\},$$

which we call prepoles of order n ; and

$$A_\infty(f) = \bigcup_{n>0} A_n(f),$$

which we call the set of prepoles.

In complex dynamics it is common to analyze the iterates of functions as a sequence in a function space. We begin by defining the notion of local uniform convergence.

DEFINITION 1.5.2. *Let X and Y be complete metric spaces. A sequence of functions $\{f_n : X \rightarrow Y\}_{n \in \mathbb{N}}$ converges locally uniformly to a function $f : X \rightarrow Y$ if on every compact subset $K \subset X$, $f_n|_K \rightarrow f$ uniformly.*

The following result can be found in [2].

PROPOSITION 1.5.3. *Let $U \subset \mathbb{C}$ be an open set. Let $f_n : U \rightarrow \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to a function f . Then f is holomorphic.*

Thus the set of holomorphic functions on U is closed in the topology of local uniform convergence.

DEFINITION 1.5.4. *Let X and Y be complete metric spaces. A family of functions $\{f : X \rightarrow Y : f \in \mathcal{F}\}$ is normal if every sequence of functions in \mathcal{F} contains a subsequence which converges locally uniformly. Furthermore, a family is normal at $x \in X$ if it is normal in a neighborhood of x .*

A normal family of functions has a property very similar to the Bolzano-Weierstrass property that says that every bounded sequence in \mathbb{R} has a convergent subsequence. One could loosely think of a normal family as a “bounded” or “restricted” collection of functions. The family of functions we consider are the iterates $\{f, f \circ f, \dots, f^n, \dots\}$ of a meromorphic function f where $f^n = f \circ f \circ \dots \circ f$ n times. The Fatou set is the set where the dynamics are predictable and well behaved. This set includes points that are attracted to a periodic cycle, for instance.

DEFINITION 1.5.5. *The Fatou set $F(f)$ of a meromorphic function $f : \mathbb{C} \rightarrow \mathbb{C}_\infty$, is defined to be*

$$F(f) = \{z : \{f^n\}_{n \in \mathbb{N}} \text{ are defined and form a normal family at } z\}.$$

If f is transcendental meromorphic then

$$F(f) = \{z \in \mathbb{C} \setminus A_\infty(f) : \{f^n\}_{n \in \mathbb{N}} \text{ forms a normal family at } z\},$$

since f^n is defined at z for all n if and only if $z \in \mathbb{C} \setminus A_\infty(f)$. In contrast to the Fatou set, the Julia set is the set where the dynamics are unpredictable and chaotic. Repelling periodic points are included in the Julia set, for example.

DEFINITION 1.5.6. *The Julia set of a meromorphic function f , $J(f)$ is defined to be $J(f) = \mathbb{C}_\infty \setminus F(f)$.*

The first key fact is that for rational maps the Fatou and Julia sets are forward and backward invariant [4]. Furthermore, the Fatou set is open, since it is a union of open sets, and so the Julia set is closed.

THEOREM 1.5.7. [4] (*Montel's Theorem*) *Suppose that a family of functions \mathcal{F} mapping $U \subset \mathbb{C}$ to \mathbb{C}_∞ has the property that $f(U) \subset \mathbb{C}_\infty \setminus \{a, b, c\}$ where a, b and c are distinct points in \mathbb{C}_∞ . Then \mathcal{F} is a normal family on U .*

Using Montel's Theorem, we have the following fact. We say a set X is *completely invariant* if $z \in X$ implies $f(z) \in X$ (unless $f(z)$ is undefined) and $f(w) = z$ implies $w \in X$.

PROPOSITION 1.5.8. [5] *The Julia set is the smallest completely invariant closed set containing at least three points.*

We recall that a transcendental meromorphic function is a meromorphic function which is not rational. These can be split into three distinct types:

- (1) $E = \{f : f \text{ is transcendental entire}\}$
- (2) $P = \{f : f \text{ is transcendental meromorphic with one pole}$
which is an omitted value}

(3) $M = \{f : f \text{ is transcendental meromorphic and not in } P \text{ or } E\}$

The family M contains functions with at least two poles or a pole that is not an omitted value. The *backward orbit* of a point z is defined as

$$O^-(z) = \{w : f^n(w) = z \text{ for some } n \in \mathbb{N}\}.$$

PROPOSITION 1.5.9. [5] *Suppose that $f \in M$. Then $\overline{A_\infty(f)} = \overline{O^-(\infty)} = J(f)$.*

We prove this in the case that f is elliptic.

PROOF. Assume that f is elliptic. We have that $\infty \in J(f)$, and so $\overline{O^-(\infty)} \subset J(f)$ by the complete invariance of $J(f)$. To show equality we only have to argue that $O^-(\infty)$ has at least three points. Since $O^-(\infty)$ contains the lattice with which f is periodic, this gives that $O^-(\infty)$ is infinite, and so has at least three points. By Montel's Theorem, we have that $\overline{O^-(\infty)} = J(f)$. \square

1.6. Periodic Points

It is often convenient to think of maps in some standard canonical form. By changing coordinates we can often simplify the map and identify qualitative features of the local (and sometimes global) dynamics.

DEFINITION 1.6.1. *Let $f, g : \mathbb{C} \rightarrow \mathbb{C}_\infty$ be meromorphic maps. If there is a homeomorphism $\phi : \mathbb{C} \rightarrow \mathbb{C}$ so that $f \circ \phi = \phi \circ g$, we say that f and g are conjugate. If ϕ is conformal, we say they are conformally conjugate.*

Suppose that $f(z_0) = z_0$ and let $\phi(z) = z - z_0$. Then set $g(z) = (\phi \circ f \circ \phi^{-1})(z)$. We see that $g(0) = 0$ and g is conformally conjugate to f . For the sake of studying local

behavior near a fixed point we make use of this observation. We remark that we may consider a fixed point to be the origin.

If ϕ has additional properties associated to it we add those. Key examples are quasi-conformal homeomorphisms and the notion of hybrid equivalences (see Section 3.3.3). The nature of fixed and periodic points can be split into three categories: attracting, repelling, and neutral. The neutral case then splits into several others depending on subtle differences in the derivative at the fixed point.

DEFINITION 1.6.2. *Suppose that f is a meromorphic function with $f^n(z_0) = z_0$, so that $f^k(z_0) \neq z_0$ for $1 \leq k \leq n-1$. We say that z_0 has period n and is attracting if $|(f^n)'(z_0)| < 1$, repelling if $|(f^n)'(z_0)| > 1$, or neutral if $|(f^n)'(z_0)| = 1$. Furthermore, we define the number $(f^n)'(z_0)$ as the multiplier of the cycle $z_0 \rightarrow f(z_0) \rightarrow \cdots \rightarrow f^{n-1}(z_0) \rightarrow z_0$.*

If $f^n(z_0) = z_0$ and $(f^n)'(z_0) = 0$, we say that z_0 is a *super-attracting periodic point* or *super-attracting cycle*. The local behavior near an attracting fixed point is surprisingly elementary. The following theorem from [4] was proved in the context of rational maps but applies to transcendental meromorphic functions as well. It relies only on local analyticity of the map near the fixed point.

THEOREM 1.6.3. *(6.3.2 in [4]) Suppose that f is analytic in a neighborhood of the origin, that $f(0) = 0$, and that $f'(0) = a$, where $0 < |a| < 1$. Then there exists a unique function g which is analytic in some disc $\{z : |z| < r\}$ with $g(0) = 0$, $g'(0) = 1$, and which satisfies $gfg^{-1}(z) = az$ for all z sufficiently close to the origin.*

This theorem states that we can locally linearize the map by changing coordinates.

The super-attracting case is only slightly different. This theorem is due to Böttcher.

THEOREM 1.6.4. *(9.1 in [41]) Suppose $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$ near the origin. There exists a local analytic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to the n^{th} power map, $w \rightarrow w^n$, throughout some neighborhood of zero. Furthermore, ϕ is unique up to multiplication by an $(n-1)^{\text{st}}$ root of unity.*

The repelling case is identical to the attracting case in that one can change coordinates to linearize the inverse. The neutral periodic points are a rich and interesting topic in their own right. The local behavior is much more complicated and, in fact, not completely understood in some cases. We begin with a case that is completely understood.

DEFINITION 1.6.5. *We say that a periodic point z_0 is rationally neutral or parabolic of period n if*

$$(f^n)'(z_0) = e^{\alpha 2\pi i}$$

and $\alpha \in \mathbb{Q}$. We say that z_0 is irrationally neutral if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Rationally neutral periodic points are always in the Julia set. This is Proposition 1.6.6. However, they form the center of a collection of “petals” in the Fatou set called *Leau petals*. To study these points we can assume that the fixed point, z_0 , is the origin, since we can conjugate by the affine map $\phi(z) = z - z_0$. For simplicity we replace f^n with f . After conjugating we have a Taylor expansion of the form

$$f(z) = az + bz^k + O(z^{k+1}),$$

where $k > 1$ and a is a root of unity, i.e. $a^l = 1$ for some integer $l \geq 1$. So

$$f^l(z) = a^l z + cz^p + O(z^{p+1}) = z + cz^p + O(z^{p+1}).$$

Let $g(z) = f^l(z)$. By Corollary 2.6.5 in [4], we have

$$g^n(z) = z + ncz^p + O(z^{p+1}).$$

This gives that $\{g^n\}_{n \in \mathbb{N}}$ is not normal at 0, and therefore $\{f^n\}_{n \in \mathbb{N}}$ is not normal at 0.

We state this more explicitly below.

PROPOSITION 1.6.6. [4] *If z_0 is a parabolic point, then $z_0 \in J(f)$.*

PROOF. Using the construction above, the p^{th} derivative of g at 0 is $(g^n)^{(p)}(0) = ncp!$, which has no convergent subsequence as n goes to infinity. \square

Leau petals are a useful tool in understanding the local behavior near a parabolic point. Various authors use different definitions of “petal.” We follow Beardon’s definition. For a slightly different approach we refer to [41].

We define a *petal of order p* to be the following:

$$\Pi_k^p(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |2k\pi/p - \theta| < \pi/p\},$$

where $0 \leq k \leq p-1$. A “flower” of Leau petals of order 3 is pictured in Figure 1.2 in addition to the Julia set of $e^{2\pi i/3}z + z^2$.

One can work harder on the above expansion near a parabolic point to obtain the following result.

LEMMA 1.6.7. (6.5.7 in [4]) *Suppose that f is analytic near 0 with*

$$f(z) = z + az^{p+1} + O(z^{p+2})$$

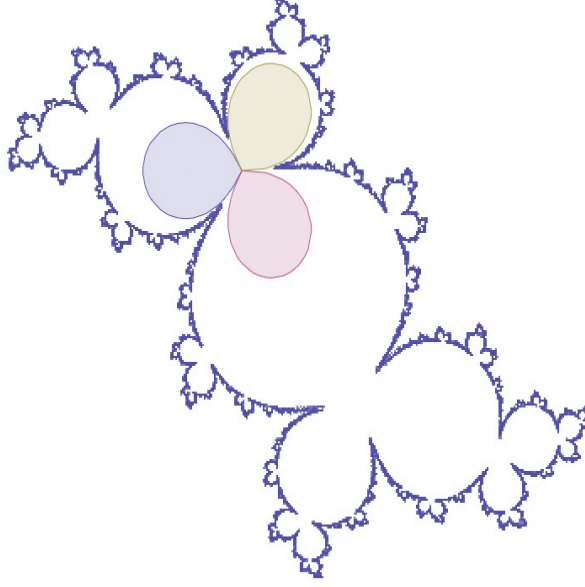


FIGURE 1.2. Leau Petals

for $a \neq 0$. Then f is conjugate near 0 to a function

$$w(z) = z - z^{p+1} + O(z^{2p+1}).$$

Using this result we can now state the local dynamical behavior near a parabolic fixed point.

THEOREM 1.6.8. (*Petal Theorem [4]*) Suppose that the analytic map f has a Taylor expansion

$$f(z) = z - z^{p+1} + O(z^{2p+1})$$

at the origin. Then for small enough t we have the following:

- (1) $f : \Pi_k^p(t) \rightarrow \Pi_k^p(t)$
- (2) $f^n \rightarrow 0$ as $n \rightarrow \infty$ on each $\Pi_k^p(t)$
- (3) $\arg(f^n(z)) \rightarrow 2k\pi/p$ locally uniformly on $\Pi_k^p(t)$ as $n \rightarrow \infty$
- (4) $|f(z)| < |z|$ on a neighborhood of the axis of each petal ($\arg(z) = 2k\pi/p$)

(5) $f : \Pi_k^p(t) \rightarrow \Pi_k^p(t)$ is conjugate to a translation

The local behavior of irrationally neutral fixed points is much more complicated than in the parabolic case and is not completely understood. Unlike the parabolic case, where each parabolic fixed point is always in the Julia set, an irrationally neutral point may be in either the Fatou set or the Julia set. If it is in the Fatou set, we have the following theorem for rational maps. We need one definition before stating the theorem.

DEFINITION 1.6.9. *A analytic map f with the following expansion near the origin, $f(z) = \lambda z + O(z^2)$ is locally linearizable if there is a map ϕ , analytic on a neighborhood of the origin, which conjugates f to the map $z \rightarrow \lambda z$ near the origin.*

THEOREM 1.6.10. [4, 41] *Let f be a rational function of degree ≥ 2 with a neutral fixed point z_0 . Then the following are equivalent:*

- (1) *f is locally linearizable around z_0*
- (2) *$z_0 \in F(f)$*
- (3) *the component $U \subset F(f)$ containing z_0 is conformally isomorphic to the open unit disk under an isomorphism which conjugates f on U to an irrational rotation of that disk.*

When this disk occurs we refer to it as a *Siegel disk* (or a *cycle of Siegel disks*) and the fixed point a *Siegel point* (respectively a *cycle of Siegel points*). (See Theorem 1.7.1)

If the map is not locally linearizable near an irrational fixed point, then that point is called a *Cremer point*. The local behavior near these points is still not well understood [4, 41].

To summarize, attracting periodic points are always in the Fatou set, and repelling periodic points are always in the Julia set. Neutral points can be in either. Considering the Julia set as the unstable set can be justified by the following theorem.

THEOREM 1.6.11. [5] *Let f be a meromorphic function. Then $J(f)$ is the closure of the set of repelling periodic points of f .*

1.7. Fatou Components and The Critical Orbit

The classification of Fatou components is an important tool for understanding the dynamics of both rational and transcendental meromorphic functions. This classification theorem is due mostly to Cremer [8] and Fatou [28, 29] and was first presented in this form by Baker, Kotus, and Lü [3].

THEOREM 1.7.1. [5] *Let f be a meromorphic function on \mathbb{C} and let U be a periodic component of $F(f)$ of period p . Then we have exactly one of the following properties:*

- (1) *U contains an attracting periodic point z_0 of period p . Then $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$, and U is called the immediate basin of attraction for z_0 .*
- (2) *∂U contains a periodic point z_0 of period p and $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$, where $g(z) = 1/(f(1/z))$.) In this case, U is a Leau domain (also called parabolic domain).*
- (3) *There exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{D}$, where \mathbb{D} is the unit disk, such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Siegel disk.*

- (4) *There exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{A}$ where A is an annulus, $A = \{z : 1 < |z| < r\}$ for some $r > 1$, such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a Herman ring.*
- (5) *There exists $z_0 \in \partial U$ such that $f^{np}(z) \rightarrow z_0$ for $z \in U$ as $n \rightarrow \infty$, but $f^p(z_0)$ is not defined. In this case, U is called a Baker domain.*

The existence of functions with Siegel discs and Herman rings is a non-trivial question. For example it was not known if Weierstrass elliptic functions have Siegel disks. We give a positive answer to this question in Section 3.7. The detection of Siegel disks and Herman rings can be done using the following characterization:

PROPOSITION 1.7.2. [5] *Let f be meromorphic and let U be a component of $F(f)$. If the closure of $\{f^n|_U\}$ (in the local uniform topology) contains non-constant functions, then U is a Siegel disk or a Herman ring.*

This is often a difficult condition to check, but it is sometimes the case that one can rule out Herman rings.

PROPOSITION 1.7.3. (Exercise 15-a in [41]) *If f is a polynomial, then f does not have any Herman rings.*

PROOF. We use the Maximum Modulus Principle. Suppose that $f^n : U \rightarrow U$ is a Herman ring. For simplicity, replace f^n with $f : U \rightarrow U$, since f^n is also a polynomial. Now, $\mathbb{C} \setminus U$ has two closed components (one can draw a Jordan curve in U separating them). One component is bounded and one is unbounded. Let U' be the union of U with the bounded component of its complement. The Maximum Modulus principle states that if $z_0 \in U$ and $|f(z)| \leq |f(z_0)|$ then f is constant. This immediately implies that

$f : U' \rightarrow U'$, since the values of f on the interior component of the complement of U cannot exceed the values of f on U , which is invariant. By analytic continuation the limit functions of $\{f^n|_U\}$ extend uniquely to limit functions of $\{f^n|_{U'}\}$. Since U' is simply connected and contains U , f cannot have any Herman rings. \square

The following result concerning elliptic functions was proved in [32].

THEOREM 1.7.4. *For any lattice Ω , \wp_Ω has no cycle of Herman rings.*

The natural question that arises is the existence of Siegel disks and Herman rings. We will not attempt to discuss the existence of Herman rings, since they do not arise in our context. We refer the reader to [42], published in 1987, which uses quasi-conformal surgery to construct examples of rational maps with Herman rings. Siegel disks were shown to exist by C. L. Siegel in 1941 [43]. We are interested in the following question:

QUESTION 1.7.5. *Do Weierstrass elliptic \wp_Ω functions have Siegel disks?*

One of the results of this thesis gives an affirmative answer to this question. In order to do this we must first show that the family $p_c(z) = z^2 + c$ has Siegel disks. Begin by conjugating to the family $\lambda z + z^2$. Let $\phi_c(z) = z - a(c)$, where $a(c) = (-1 + \sqrt{1 - 4c})/2$. Thus $(\phi_c^{-1} \circ p_c \circ \phi_c)(z) =$

$$(z - a)^2 + c + a = z^2 - 2az + a^2 + c + a = -2az + z^2 = \lambda(a(c))z + z^2.$$

Consider $\lambda = e^{2\pi i\alpha}$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Since α is irrational it has an infinite continued fraction expansion

$$\alpha = [a_0; a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}}},$$

and so we let $p_n/q_n = [0; a_1, a_2, a_3, \dots, a_n]$ be the n^{th} rational approximation. If the series

$$(1.17) \quad \sum_{n=1}^{\infty} \frac{\log(q_{n+1})}{q_n}$$

converges, then we say that α is a *Brjuno number* (also spelled Bryuno). The following theorem is due to Brjuno, Rüssmann, and Yoccoz.

THEOREM 1.7.6. (*Theorems 11.10 and 11.11 in [41]*) *If α is a Brjuno number, then for $\lambda = e^{2\pi i\alpha}$, $f(z) = \lambda z + O(z^2)$ is locally linearizable at the origin. Conversely, if λ is not a Brjuno number, then $\lambda z + z^2$ is not linearizable at the origin.*

As a consequence of this and Proposition 1.7.3 we have the following.

PROPOSITION 1.7.7. *If λ is a Brjuno number then $\lambda z + z^2$ has a Siegel disk at the origin.*

For the sake of illustration we present one example of a Brjuno number, the golden mean, $\phi = (\sqrt{5} + 1)/2$. Since ϕ is a fixed point of $1 + 1/z$ we have that $\phi = [1; 1, 1, 1, \dots]$

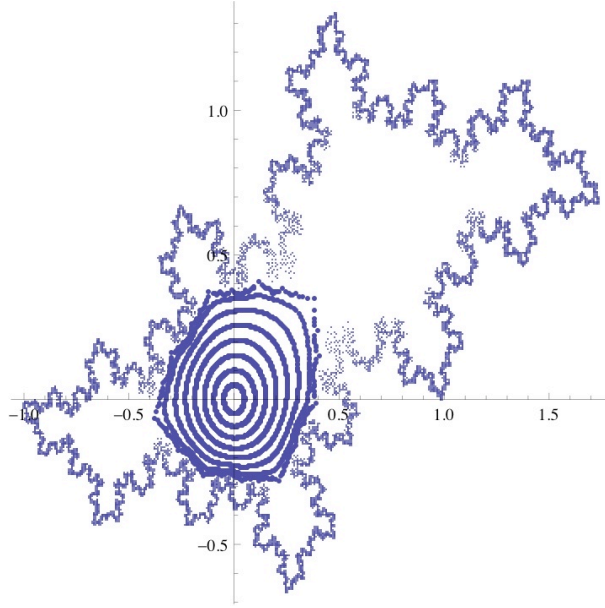


FIGURE 1.3. Julia set and invariant circles for $e^{2\pi i \phi} z + z^2$

and $\phi - 1 = [0; 1, 1, 1, \dots]$. It is easy to check that $p_n/q_n = F_{n+1}/F_n$, where F_n are the Fibonacci numbers. We have that

$$\frac{\log q_{n+1}}{q_n} = \frac{\log F_{n+1}}{F_n}.$$

Doing a ratio test one can check that $\left(\frac{\log F_{n+2}}{F_{n+1}}\right) / \left(\frac{\log F_{n+1}}{F_n}\right) \rightarrow 1/\phi < 1$ and so the summation converges. Thus, the golden ratio is a Brjuno number.

EXAMPLE 1.7.8. *Let ϕ be the golden ratio. Then*

$$e^{2\pi i \phi} z + z^2$$

has a Siegel disk at the origin. A numerical attempt to show a Julia set and invariant circles for this function is made in Figure 1.3

The existence of Fatou components that are not pre-periodic is an important issue. These components that are not periodic or pre-periodic are called “wandering domains.” Sullivan settled the question for rational maps.

THEOREM 1.7.9. [4] [Sullivan] *If f is rational, then f has no wandering domains, i.e., all Fatou components are pre-periodic.*

This result is not true for transcendental meromorphic functions. The first example was given by Baker and others have followed. One example is $f(z) = z - 1 + e^{-z} + 2\pi i$ [5, 3]. To see that this has a wandering domain, consider Newton’s method applied to $h(z) = e^z - 1$. The function that arises is $g(z) = z - h'(z)/h(z) = z - 1 + e^{-z}$. Baker showed that $J(f) = J(g)$. The points $z_k = 2k\pi i$ are super-attracting fixed points of g . Each has a simply connected basin of attraction U_k . These basins of attraction of g are wandering domains for f [3].

There are classes of meromorphic functions which do not have wandering domains. These classes include a class called S. Class S functions have only finitely many critical and asymptotic values. The reader will notice that both f and g as above have infinitely many critical values.

DEFINITION 1.7.10. *A complex number α is called an asymptotic value of a transcendental meromorphic function $f : \mathbb{C} \rightarrow \mathbb{C}_\infty$ if there is a path $s : [0, 1) \rightarrow \mathbb{C}$ with the property*

$$\lim_{t \rightarrow 1} s(t) = \infty$$

so that

$$\lim_{t \rightarrow 1} f(s(t)) = \alpha.$$

A good example of a function with asymptotic values is e^z . Taking curves along the real line, we can see that both 0 and ∞ are asymptotic values.

DEFINITION 1.7.11. *A function is of class S if it has finitely many critical values and finitely many asymptotic values.*

PROPOSITION 1.7.12. *An elliptic function f has no asymptotic values.*

PROOF. Assume that an elliptic function f on the lattice $\Omega = \langle \omega_1, \omega_2 \rangle$ has an asymptotic value α . Let $\epsilon > 0$. We have a path $s : [0, 1) \rightarrow \mathbb{C}$ with the property

$$\lim_{t \rightarrow 1} s(t) = \infty,$$

so that

$$\lim_{t \rightarrow 1} f(s(t)) = \alpha.$$

So there exists $\delta > 0$ so that if $1 - t' < \delta$ then $|f(s(t')) - \alpha| < \epsilon$. First, $s : (1 - \delta, 1) \rightarrow \mathbb{C}$ must pass through infinitely many period parallelograms. If it passed through only finitely many period parallelograms then the path could not approach infinity. Consider the quotient map

$$pr : \mathbb{C} \rightarrow \mathbb{C}/\Omega,$$

which is continuous. By periodicity we have that $f(s(t')) = f(pr(s(t')))$. Thus we may consider the curve $pr(s(t))$ as a subset of a period parallelogram, say

$$P_\Omega = \{t\omega_1 + s\omega_2 : 0 \leq t, s < 1\}.$$

Since $s : (1 - \delta, 1) \rightarrow \mathbb{C}$ passes through infinitely many period parallelograms, we have that $pr(s(t'))$ has infinitely many values on $\{t\omega_1 : 0 \leq t < 1\} \cup \{s\omega_2 : 0 \leq s < 1\}$. Using

the order on $[1 - \delta, 1)$, this produces a sequence $\{z_n\}$, and so evidently

$$\lim_{t \rightarrow 1} f(s(t)) = \lim_{n \rightarrow \infty} f(z_n) = \alpha.$$

Now at least one of $\{t\omega_1 : 0 \leq t < 1\}$ or $\{s\omega_2 : 0 \leq s < 1\}$ contains an infinite subsequence $\{z_{n_k}\}$. For simplicity assume it is $\{t\omega_1 : 0 \leq t < 1\}$. This establishes that it is sufficient to look along the ray $\{t\omega_1 : 0 \leq t < \infty\}$ for the asymptotic value α . But since f is periodic along this ray it cannot be the case that $|f(z) - \alpha| < \epsilon$ for large values of t in $\{t\omega_1 : 0 \leq t < \infty\}$. \square

THEOREM 1.7.13. [5] *A meromorphic function of class S has no wandering domains.*

PROPOSITION 1.7.14. *An elliptic function is of class S and so has no wandering domains or Baker domains.*

PROOF. We have already established that an elliptic function has only finitely many critical values in Corollary 1.3.2. Since an elliptic function is doubly periodic, it cannot have any asymptotic values. Thus it is class S and cannot have wandering domains [5] or Baker domains [27]. \square

CHAPTER 2

Connectivity of Julia Sets of the Weierstrass Elliptic Functions on a Square Lattice

In this chapter we prove that the Julia set of $\wp_\lambda = \wp_{\langle\lambda, \lambda i\rangle}$ is connected for all parameter values $\lambda \in \mathbb{C}^*$. We begin by discussing a sufficient condition for the Julia set of a Weierstrass elliptic function to be connected. Using this tool we first prove the result in the case when the parameter is real. We then generalize the result for all parameter values. It is sufficient for us to prove the result when \wp_λ has an attracting or parabolic cycle. It has already been proved in [33] that $J(\wp_\lambda)$ is connected when \wp_λ has a cycle of Siegel disks. If \wp_λ does not have a non-repelling cycle, then $J(\wp_\lambda)$ is the whole Riemann sphere and thus is connected.

2.1. General Properties of Julia Sets of Elliptic Functions

To determine if a subset of \mathbb{C}_∞ is connected we have the following fundamental proposition.

PROPOSITION 2.1.1. [4] *A subset $S \subset \mathbb{C}_\infty$ is connected if and only if $\mathbb{C}_\infty \setminus S$ is simply connected, i.e. has simply connected components.*

Applying this to Julia sets we see that a Julia set is connected if and only if the Fatou set is simply connected. The following theorem, adapted from [40, Theorem 3.1], is proved in [33]. We present a similar proof here.

THEOREM 2.1.2. [33] *If every Fatou component of $F(\wp_\Omega)$ contains 0 or 1 critical values, then $J(\wp_\Omega)$ is connected.*

PROOF. If the Fatou set is empty, then $J(\wp_\Omega) = \mathbb{C}_\infty$, which is connected and we are done. Assume that the Fatou set is non-empty. It must be shown that $F(\wp_\Omega)$ is simply connected. We show this is the case component-wise. Let $V \subset F(\wp_\omega)$ be a component. Let α be a loop in V . Since \wp_Ω has no wandering domains or Baker domains (1.7.14) and no Herman rings (1.7.4), V must be pre-periodic. Thus there is a large enough k so that $\wp_\Omega^k(\alpha)$ is contained in a simply connected open set $U \subset V$ in:

- (1) an immediate basin of attraction for an attracting or super-attracting periodic point;
- (2) a Leau domain for a periodic parabolic point;
- (3) or a Siegel disk in a cycle of Siegel disks.

It was shown in [33] that a Siegel disk is contained in a period parallelogram. Thus by choosing a large enough k and small enough U we may assume that U lies in a period parallelogram P so that $\overline{U} \subset P$. We may also assume that the boundary of U contains no critical points. We need only to show that $\wp_\Omega^{-k}(U)$ is simply connected. We use induction on k . The $k = 0$ case is established, since U is simply connected. Assume that $\wp_\Omega^{k-1}(U)$ is simply connected. Let W be a component of $\wp_\Omega^{k-1}(U)$, which we have assumed is simply connected. Suppose that W contains no critical value. Then each fundamental region $\wp_\Omega^{-1}(W)$ consists of two disjoint simply connected regions. If W contains one critical value then $\wp_\Omega^{-1}(W)$ consists of one simply connected region in each fundamental region. This is because $\wp_\Omega : \wp_\Omega^{-1}(W) \rightarrow W$ is a two-to-one branched cover in each fundamental region. Applying this argument to each component, we have our result. \square

The connectivity of $J(\wp_\Omega)$ when Ω is triangular ($\Omega = \langle \lambda, e^{\pi i/3} \lambda \rangle$) was settled in [33] and [32].

THEOREM 2.1.3. [33] *If Ω is triangular then $J(\wp_\Omega)$ is connected.*

PROOF. It was shown in [32] that the critical values of \wp_Ω lie in different components of $F(\wp_\Omega)$, and so Theorem 2.1.2 applies to give the result. \square

The following classification of Fatou components for Weierstrass elliptic functions with square lattices was established in [32]. First we define the general basin of attraction for an attracting or parabolic periodic cycle. We recall that we defined the immediate basin in 1.7.1.

DEFINITION 2.1.4. *Let $\{p_1, \dots, p_k\}$ be an attracting or parabolic cycle of period k for the meromorphic function f . We define the basin of attraction to be the set of points z so that $f^{nk}(z) \rightarrow p_i$ for some $1 \leq i \leq k$.*

Whereas the immediate basin is the minimal collection of Fatou components containing the cycle, the basin is the set of all pre-images of the immediate basin.

THEOREM 2.1.5. [32] *Let Ω be a square lattice. Then exactly one of the following must occur:*

- (1) $F(\wp_\Omega)$ is empty and $J(\wp_\Omega) = \mathbb{C}_\infty$;
- (2) $F(\wp_\Omega)$ is the basin of an attracting cycle;
- (3) $F(\wp_\Omega)$ is the basin of a parabolic cycle;
- (4) or $F(\wp_\Omega)$ consists of a cycle of Siegel disks and their pre-images.

2.2. Real Rectangular Square Lattices with an Attracting Fixed Point

Our goal will be to prove the following theorem in the next section.

THEOREM 2.2.1. *For each $\lambda \in \mathbb{C}^*$ the Julia set of $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$ is connected.*

We say that a lattice Ω is *real rectangular* if there are real numbers $a > 0$ and $b > 0$ so that $\Omega = \langle a, bi \rangle$. Let $\Omega(\lambda) = \langle \lambda, \lambda i \rangle$ for $\lambda \neq 0$. In this section we assume that $\lambda > 0$ to illustrate the technique for obtaining the general result. Define $\wp_\lambda = \wp_{\Omega(\lambda)}$. Since $i\Omega(\lambda) = \Omega(i\lambda) = \Omega(\lambda)$, the homogeneity property in Equation (1.5) gives

$$\wp_\lambda(iz) = \wp_{i\lambda}(iz) = i^{-2}\wp_\lambda(z) = -\wp_\lambda(z).$$

We define $\mathbb{R}^+ = \{x > 0 : x \in \mathbb{R}\}$ and $\mathbb{R}^- = \{x < 0 : x \in \mathbb{R}\}$. By Theorem 1.2.10 and Proposition 1.2.11 with (1.5) we have:

$$(2.1) \quad \wp_\lambda : \mathbb{R} \setminus \Omega(\lambda) \rightarrow \mathbb{R}^+$$

$$(2.2) \quad \wp_\lambda : i\mathbb{R} \setminus \Omega(\lambda) \rightarrow \mathbb{R}^-.$$

THEOREM 2.2.2. *Suppose Γ is a real rectangular square lattice and that \wp_Γ has an attracting fixed point a_0 . Then $J(\wp_\Gamma)$ is connected.*

PROOF. Let $\Gamma = \langle \lambda, \lambda i \rangle$ for $\lambda > 0$. From Propositions 1.2.9 and 1.2.12 we have that $e_1 = \wp_\lambda(\lambda/2) > 0$, $e_2 = -e_1$ and $e_3 = 0$. It is sufficient to show that each Fatou component contains at most one critical value, by Theorem 2.1.2. The critical value $e_3 = 0$ is not in the Fatou set, since it is a pole. We proceed by contradiction.

Suppose that a Fatou component, U , contains both e_1 and $e_2 = -e_1$. Each Fatou component is path connected since it is open and connected. Consider a path C connecting e_1 and $-e_1$ in U .

We claim that $C \cap (i\mathbb{R} \setminus 0) \neq \emptyset$ since C cannot contain 0. To prove this claim we parameterize C on $[0, 1]$. Set $\gamma(t) = r(t)e^{i\theta(t)}$, where $\gamma : [0, 1] \rightarrow C$ so that $\gamma(0) = e_1$ and $\gamma(1) = -e_1$. $r(t)$ is a non-vanishing function since $0 \notin C$. Since $e_1 \in \mathbb{R}$, we have that $\theta(0) = 0$ and $\theta(1) = (2k + 1)\pi$ for some $k \in \mathbb{Z}$. Since θ is a continuous function of t , the Intermediate Value Theorem applies, giving that θ attains the value $\pi/2$ or $-\pi/2$ for some value $t_0 \in [0, 1]$. We have that $\gamma(t_0) = r(t_0)e^{i\pi/2}$ or $\gamma(t_0) = r(t_0)e^{-i\pi/2}$ for some t_0 . This establishes that $C \cap (i\mathbb{R} \setminus 0) \neq \emptyset$.

We note that C is a compact subset of the immediate basin of attraction of the attracting fixed point a_0 , which is necessarily real and positive since the critical orbit, $\{\varphi_\lambda^n(e_1)\} = \{\varphi_\lambda^n(e_2)\}$, is contained in the positive real axis and cannot cluster on zero. All iterates of φ_Γ are defined on C since all pre-poles lie in the Julia set. Now $\{\varphi_\Gamma^n\}_{n \in \mathbb{N}}$ converges uniformly on C to the constant function a_0 in the Euclidean metric [5, 4]. So for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $\sup_{z \in C} |\varphi_\Gamma^n(z) - a_0| < \epsilon$.

We claim that for each $n \in \mathbb{N}$, $\varphi_\Gamma^n(C)$ contains a point in each of \mathbb{R}^+ , \mathbb{R}^- , and $i\mathbb{R} \setminus 0$ and is connected. We proceed by induction. We have established these properties for C (the $n = 0$ case). For the induction hypothesis we assume that $\varphi_\Gamma^n(C)$ contains a point in each of \mathbb{R}^+ , \mathbb{R}^- , and $i\mathbb{R} \setminus 0$ and is connected. We can immediately establish that $\varphi_\Gamma^{n+1}(C)$ is connected since all iterates are defined on C and φ_Γ is continuous away from its pre-poles. Furthermore, $\varphi_\Gamma^{n+1}(C)$ contains a point on \mathbb{R}^+ and a point on \mathbb{R}^- by equations (2.1) and (2.2). Lastly $\varphi_\Gamma^{n+1}(C)$ contains a point on $i\mathbb{R} \setminus 0$ by the same argument given for C ($n = 0$ case), replacing e_1 and $-e_1$ by the point on \mathbb{R}^+ and a point on \mathbb{R}^- respectively.

Choose a point a_n in $\wp_{\Gamma}^n(C) \cap \mathbb{R}^-$ for each $n \in \mathbb{N}$. We have that

$$\sup_{z \in C} |\wp_{\Gamma}^n(z) - a_0| \geq |a_n - a_0| > a_0$$

for all $n \in \mathbb{N}$, since $a_n < 0$ and $a_0 > 0$. This contradicts the uniform convergence of \wp_{Γ}^n on C to the constant function a_0 . Thus every Fatou component contains at most one critical value, and so the Julia set is connected. \square

2.3. Proof of Theorem 2.2.1

We now return to the general case. Consider $\lambda \in \mathbb{C}$ and set $\wp_{\lambda} = \wp_{\langle \lambda, \lambda i \rangle}$ as before.

DEFINITION 2.3.1. *Fix $\lambda \in \mathbb{C}^*$. Define $L_1 = \{t\lambda^{-2} : t \in \mathbb{R}^+\}$ and $L_2 = -L_1$. Furthermore, define $S_1 = \{t\lambda : t \in \mathbb{R} \setminus \mathbb{Z}\}$. Set $S_2 = iS_1$.*

PROPOSITION 2.3.2. *For each $\lambda \in \mathbb{C}^*$ we have that*

$$\wp_{\lambda}(S_1) \subset L_1$$

and

$$\wp_{\lambda}(S_2) \subset L_2.$$

PROOF. If $z \in S_1$, then $z = t\lambda$ for some $t \in \mathbb{R} \setminus \mathbb{Z}$. By the homogeneity equations (1.5), we have that

$$\wp_{\lambda}(t\lambda) = \lambda^{-2}\wp_1(t).$$

Furthermore, $\wp_1(t) > 0$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$ by Equation (2.1), giving that $\lambda^{-2}\wp_1(t) \in L_1$.

If $z \in S_2$, then $z = itk$ for $t \in \mathbb{R} \setminus \mathbb{Z}$. We have that $i\langle \lambda, \lambda i \rangle = \langle \lambda, \lambda i \rangle$. Using the homogeneity equations a second time, we have

$$\wp_{\lambda}(it\lambda) = (i\lambda)^{-2}\wp_1(t) = -\lambda^{-2}\wp_1(t).$$

Since $\wp_1(t) > 0$ for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have that $-\lambda^{-2}\wp_1(t) \in L_2$. \square

We observe that $e_1 \in L_1$ and $e_2 \in L_2$ by equation (1.15).

LEMMA 2.3.3. *Fix $\lambda \in \mathbb{C}^*$. Let $p \neq 0$. Then*

$$K(p) = \max\left\{\inf_{z \in L_1} d(z, p), \inf_{z \in L_2} d(z, p)\right\} > 0,$$

where d is the Euclidean metric on \mathbb{C} .

PROOF. Suppose that $p \neq 0$. If $p \in L_1$ then

$$K(p) = \inf_{z \in L_2} d(z, p) \geq d(0, p) > 0.$$

Similarly for $p \in L_2$. Suppose that $p \notin L_1 \cup L_2 \cup \{0\}$. Denote $L = L_1 \cup L_2 \cup \{0\}$ which is a line passing through the origin. The distance c from p to L is positive, since p is not on L . By the triangle inequality, we have $K(p) \geq c > 0$. Thus $K(p) > 0$. \square

The distance $K(p)$ depends on λ , but we write $K(\lambda, p) = K(p)$ as in Lemma 2.3.3.

LEMMA 2.3.4. *Let C be a curve connecting $z_1 \in L_1$ and $z_2 \in L_2$ in \mathbb{C} . Let H be a line passing through the origin. Then $C \cap H \neq \emptyset$.*

PROOF. Parameterize C on $[0, 1]$ by $\gamma : [0, 1] \rightarrow C$. If C contains the origin then we are done. Suppose that C does not contain the origin. Let $\gamma(t) = r(t)e^{i\theta(t)}$. Let $\theta_0 = \theta(0)$ and $\theta_1 = \theta(1) = \theta_0 + (2k + 1)\pi$, where $k \in \mathbb{Z}$. If $w \in H$ then $\arg(w) = \theta_H$ or $\arg(w) = \theta_H - \pi$ for some $\theta_H \in (0, \pi]$, and H contains all such w 's. We have that θ is a continuous function of t , so the Intermediate Value Theorem gives that $\theta(t_0) = \theta_H$, or $\theta(t_0) = \theta_H - \pi$, for some $t_0 \in [0, 1]$. Thus, there is $t_0 \in [0, 1]$ such that $\gamma(t_0) = r(t_0)e^{i\theta(t_0)} \in H$. \square

THEOREM 2.3.5. *Suppose that \wp_λ has an attracting fixed point p . Then $J(\wp_\lambda)$ is connected.*

PROOF. It is sufficient to show that each Fatou component contains at most one critical value [33]. The critical value 0 is not in the Fatou set, since it is a pole. We proceed by contradiction.

Suppose that a Fatou component U contains both e_1 and $e_2 = -e_1$. It is evident that U must be the immediate basin of the attracting fixed point, since the immediate basin must contain a critical point [4, 5], and e_1 and $-e_1$ are both in U . Each Fatou component is path connected, since it is open and connected. Let C be a curve connecting e_1 and $-e_1$ in U .

All iterates of \wp_k are defined on C , since all pre-poles lie in the Julia set. Since C is a compact subset of the immediate basin, $\{\wp_\lambda^n\}_{n \in \mathbb{N}}$ converges uniformly on C to the constant function p in the Euclidean metric [4, 5]. So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $\sup_{z \in C} |\wp_\lambda^n(z) - p| < \epsilon$.

We claim that for each $n \in \mathbb{N}$, $\wp_k^n(C)$ contains a point on each of L_1 , L_2 , S_1 , and S_2 and is connected. We proceed by induction. We first establish these properties for the $n = 0$ case, C .

By construction C contains a point on L_1 and on L_2 . We establish that $C \cap S_i \neq \emptyset$ for $i = 1, 2$. Fix i . Now $\overline{S_i}$ is a line passing through the origin, so by Lemma 2.3.4 we have that $C \cap \overline{S_i} \neq \emptyset$. Since C does not contain any poles, we have that $C \cap S_i \neq \emptyset$.

For the induction hypothesis, we assume that $\wp_\lambda^n(C)$ contains a point in L_1 , L_2 , S_1 , and S_2 and is connected. We can immediately establish that $\wp_\lambda^{n+1}(C)$ is connected, since all iterates are defined on C and \wp_k is continuous off of its pre-poles. Furthermore,

$\wp_\lambda^{n+1}(C)$ contains a point on L_1 and a point on L_2 by Proposition 2.3.2. Lastly, we can establish that $\wp_\lambda^{n+1}(C)$ contains a point on S_1 and S_2 by Lemma 2.3.4.

If $p \notin L_2$ choose a_n in $\wp_\lambda^n(C) \cap L_2$ for each $n \in \mathbb{N}$ (if $p \in L_2$ then choose a_n in L_1).

We have that

$$\sup_{z \in C} |\wp_\lambda^n(z) - p| \geq |a_n - p| > K(p)$$

for all $n \in \mathbb{N}$, by Lemma 2.3.3. This contradicts the uniform convergence of \wp_λ^n on C to the constant function p . Thus every Fatou component contains at most one critical value, and so the Julia set is connected. \square

We now generalize this result to the case when \wp_λ has an attracting or parabolic cycle. If \wp_λ has a parabolic cycle $\{p_0, \dots, p_{l-1}\}$, then $\wp_\lambda^{ln} \rightarrow p_j$ locally uniformly on the immediate basin of p_j [5]. Furthermore, the immediate basin of a parabolic cycle contains a critical point [5]. We have the following theorem. The proof utilizes the same tools that were used in the above result.

THEOREM 2.3.6. *Suppose that \wp_λ has an attracting or parabolic cycle $\{p_0, \dots, p_{l-1}\}$. Then $J(\wp_\lambda)$ is connected.*

PROOF. Again, it is sufficient to show that each Fatou component contains at most one critical value [33]. The critical value 0 is not in the Fatou set, since it is a pole. We proceed by contradiction.

Suppose that a Fatou component U contains both e_1 and $e_2 = -e_1$. It is evident that U must be the immediate basin of the attracting cycle, since the immediate basin must contain a critical point [4, 5], and e_1 and $-e_1$ are both in U . Each Fatou component is path connected, since it is open and connected. Let C be a curve connecting e_1 and $-e_1$ in U .

All iterates of \wp_λ are defined on C since all pre-poles lie in the Julia set. Since C is a compact subset of the immediate basin, $\{\wp_\lambda^{ln}\}_{n \in \mathbb{N}}$ converges uniformly on C to the constant function p_j in the Euclidean metric for some $0 \leq j \leq l-1$ [4, 5]. This means that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $\sup_{z \in C} |\wp_\lambda^{ln}(z) - p_j| < \epsilon$ for $n > N$.

We have established in Theorem 2.3.5 that for each $m \in \mathbb{N}$, $\wp_\lambda^m(C)$ contains a point on each of L_1 , L_2 , S_1 , and S_2 and is connected.

If $p_j \notin L_2$, choose a_n in $\wp_\lambda^{ln}(C) \cap L_2$ for each $n \in \mathbb{N}$ (if $p_j \in L_2$ then choose a_n in L_1).

We have that

$$\sup_{z \in C} |\wp_\lambda^{ln}(z) - p_j| \geq |a_n - p_j| > K(p_j)$$

for all $n \in \mathbb{N}$, by Lemma 2.3.3. This contradicts the uniform convergence of $\{\wp_\lambda^{ln}\}_{n \in \mathbb{N}}$ on C to the constant function p_j . Thus every Fatou component contains at most one critical value, and so the Julia set is connected.

□

We now prove the main result for all square lattices.

PROOF. (Proof of Theorem 2.2.1) If $F(\wp)$ is empty, then $J(\wp) = \mathbb{C}_\infty$ and so is connected. (For example, this occurs when the lattice is a so-called “rhombic square” lattice [30]. It also occurs when the critical orbit lands on a pole. There are other examples as well; n.b. [31]). If \wp has an attracting cycle or a parabolic cycle, we have that $J(\wp)$ is connected by Theorems 2.3.6 and ???. If the Fatou set consists of pre-periodic Siegel disks, then the Julia set is connected [5, 33]. This exhausts all possible Fatou components. Thus $J(\wp_\lambda)$ is connected for all $\lambda \neq 0$.

□

CHAPTER 3

Mandelbrot-Like Sets in Parameter Space

In this chapter we show that the parameter space of $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$, parameterized by the lattice generator λ , contains infinitely many homeomorphic copies of the Mandelbrot set which are pairwise dynamically inequivalent up to a conformal change of coordinates. We use the tools of Douady and Hubbard [25] concerning polynomial-like mappings to prove the result. We begin by discussing the parameterization of iterated Weierstrass elliptic \wp functions as presented by J. Hawkins and L. Koss in [32]. We then proceed to construct analytic families of quadratic-like mappings corresponding to each Mandelbrot-like set. Then, using a theorem of Douady and Hubbard, we show that the Mandelbrot-like sets are actually homeomorphic to the Mandelbrot set.

3.1. Holomorphic Families

The theory of holomorphic families of meromorphic maps of the sphere was first introduced by Mañé, Sad, and Sullivan in [38]; McMullen and Sullivan furthered this study in [39]. The setting for this theory was extended to meromorphic maps of class S by Keen and Kotus in [36] and adapted to \wp_λ by Hawkins and Koss in [32].

DEFINITION 3.1.1. (1) *A holomorphic family of meromorphic maps f_λ over a complex manifold M is a holomorphic map $M \times \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$, given by $(\lambda, z) \mapsto f_\lambda(z)$.*

- (2) $M^{top} \subset M$ is the set of points λ that have a neighborhood U with the property that $\omega \in U$ implies there is a homeomorphism $\phi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ such that $f_\omega = \phi^{-1} \circ f_\lambda \circ \phi$.
- (3) The set $M^{qc} \subset M$ is defined similarly except that ϕ must be quasiconformal.
- (4) Suppose that f_λ has a finite singular set for every $\lambda \in M$. Let $M_0 \subset M$ be the set of parameters such that the number of singular values of f_λ is locally constant. For $\lambda \in M_0$, the singular values can be locally labelled by holomorphic functions $s_1(\lambda), s_2(\lambda), \dots, s_n(\lambda)$. A singular orbit relation is a set of integers (i, j, m, n) with $m, n \geq 0$ such that $f_\lambda^m(s_i(\lambda)) = f_\lambda^n(s_j(\lambda))$. The set $M^{post} \subset M_0$ of postsingularly stable parameters consists of all λ such that the set of singular orbit relations is locally constant.
- (5) A holomorphic motion of a set $J \subset \mathbb{C}_\infty$ over a connected complex manifold with basepoint (M, λ_0) is a map $\phi : M \times J \rightarrow \mathbb{C}_\infty$ given by $(\lambda, z) \mapsto \phi_\lambda(z)$ satisfying
- (a) for each fixed $z \in J$, $\phi_\lambda(z)$ is holomorphic in λ ;
 - (b) for each fixed λ , $\phi_\lambda(z)$ is an injective function of z ;
 - (c) $\phi_{\lambda_0}(z) = z$ for all $z \in J$; i.e., it is the identity function at the basepoint.
- (6) A holomorphic motion over (M, λ_0) respects the dynamics if

$$\phi_\lambda(f_{\lambda_0}(z)) = f_\lambda(\phi_\lambda(z))$$

whenever z and $f_{\lambda_0}(z)$ both belong to J .

- (7) The set $M^{stab} \subset M$ denotes the J -stable set of parameters such that the Julia set moves by a holomorphic motion respecting the dynamics.

The next result was proved in [36] using the techniques in [39].

THEOREM 3.1.2. *For any holomorphic family of meromorphic maps of class S defined over the complex manifold M :*

- (1) *The topologically stable parameters are open and dense.*
- (2) $M^{qc} = M^{post} = M^{top}$;
- (3) M^{stab} *is the set of parameters for which the total number of attracting and superattracting cycles of f_λ is constant in a neighborhood of λ ;*
- (4) M^{stab} *is open and dense in M .*

Often one may study a smaller set of parameters to obtain information about the whole parameter space.

DEFINITION 3.1.3. *We will say that a parameter space M is reduced if $\lambda, \lambda' \in M$ and $\lambda \neq \lambda'$ implies that f_λ is not conformally conjugate to $f_{\lambda'}$.*

In our setting we are interested in parameterizing Weierstrass elliptic functions on square lattices. It was shown in [32] that the parameterization $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$ where $\lambda \in \mathbb{C}^*$ forms a holomorphic family. We present the relevant results here.

PROPOSITION 3.1.4. [32, Proposition 6.2] *Let $\psi(z) = e^{2\pi i/3}z$. Then*

$$\psi \circ \wp_\lambda = \wp_{e^{\pi i/6}\lambda} \circ \psi.$$

PROOF. Plugging in and using the homogeneity equations (1.5), we have

$$(\psi \circ \wp_\lambda)(z) = e^{2\pi i/3} \wp_\lambda(z) = \frac{1}{(e^{\pi i/6})^2} \wp_\lambda(z) = \wp_{e^{\pi i/6}\lambda}(e^{\pi i/6}\lambda),$$

which completes the proof. □

The converse is also true.

THEOREM 3.1.5. *The maps \wp_λ and \wp'_λ are conformally conjugate if and only if $\lambda = e^{n\pi i/6}\lambda'$ for some $n \in \mathbb{Z}$.*

We begin with a lemma.

LEMMA 3.1.6. *For $\lambda, \lambda' \in \mathbb{C}^*$ the maps \wp_λ and $\wp_{\lambda'}$ are identically equal if and only if $\lambda' = i^k \lambda$ for some $k \in \mathbb{Z}$*

PROOF. From Proposition 1.1.2 we have that the lattices $\langle \lambda, \lambda i \rangle$ and $\langle \lambda', \lambda' i \rangle$ are equivalent if and only if

$$\begin{pmatrix} \lambda' \\ \lambda' i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda i \end{pmatrix}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. By factoring out λ and λ' , we can obtain the following:

$$\frac{\lambda'}{\lambda} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Let $r = \frac{\lambda'}{\lambda}$ and break r into real and imaginary parts, $r = r_1 + ir_2$. From this we obtain

$$\begin{pmatrix} r_1 + ir_2 \\ ir_1 - r_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

, which implies that $a = r_1$, $b = r_2$, $c = -r_2$, and $d = r_1$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have that $r_1^2 + r_2^2 = 1$ and r_1 and r_2 are integer valued. Thus $r = \frac{\lambda'}{\lambda} = 1, -1, i$, or $-i$.

□

PROOF. (of Theorem 3.1.6) (\Rightarrow) Suppose that \wp_λ and \wp'_λ are conformally conjugate. By Lemma 6.3 in [32], we have that the conformal conjugacy is of the form $\psi(z) = az$. This induces the following equation:

$$(3.1) \quad \wp_\lambda(az) = a\wp_{\lambda'}(z).$$

Using homogeneity, we obtain

$$(3.2) \quad \frac{1}{a^2}\wp_{\lambda/a}(z) = a\wp_{\lambda'}(z).$$

Since both $\wp_{\lambda/a}$ and $\wp_{\lambda'}$ share the same poles, the lattices they are defined on are identical. This implies that $\wp_{\lambda/a} = \wp_{\lambda'}$, and so $a^3 = 1$. Thus, a is a third root of unity.

Using Lemma 3.1.6 we have that $\lambda/a = i^k \lambda'$, i.e., $\lambda = ai^k \lambda'$ for some $k \in \mathbb{Z}$. We must show that $ai^k = e^{n\pi i/6}$ for some n . Since $a = e^{2l\pi i/3}$ for some $l \in \mathbb{Z}$, and $i = e^{\pi i/2}$, we have

$$ai^k = e^{n\pi i/6} e^{k\pi i/2} = e^{(n+3k)\pi i/6},$$

which gives the first direction of the proof.

(\Leftarrow) If $\lambda = e^{n\pi i/6} \lambda'$ then set $\phi(z) = e^{2\pi i/3} z$. Using Proposition 3.1.4, we have that $\phi \circ \wp_\lambda = \wp_{\lambda'} \circ \phi$. □

We define

$$(3.3) \quad M = \mathbb{C}^* / \sim,$$

where $\lambda_1 \sim \lambda_2$ if $\lambda_1 = e^{\pi i/6} \lambda_2$. The following result was proved in [32].

PROPOSITION 3.1.7. [32, Corollary 6.6 and Proposition 6.3] *The family $\wp_\lambda = \wp_{\langle \lambda, \lambda i \rangle}$ with $\lambda \in M$ forms a reduced holomorphic family of maps over M . Furthermore, $M^{stab} = M^{qc}$ and forms an open dense set in M .*

3.2. Analytic Families of Quadratic-Like Mappings

We make heavy use of a conformally conjugate family of mappings $W_\mu(z) = \mu \wp_\gamma(\gamma z)$, each of which is conformally conjugate to a unique \wp_λ .

A Simplifying Family of Elliptic Functions.

In this subsection we show that we can think of \wp_λ as a multiple of \wp_1 up to conformal conjugacy.

DEFINITION 3.2.1. *For $\mu \in \mathbb{C}^*$ define*

$$(3.4) \quad W_\mu(z) = \mu \wp_\gamma(\gamma z) = \frac{\mu}{\gamma^2} \wp_1(z),$$

where γ is the lemniscate constant defined as the positive root of $\gamma^2 = \wp_1(1/2)$.

We observe that $W_\mu(z) = (\mu/\gamma^2) \wp_1(z)$, which is a complex parameter times the Weierstrass elliptic function on the lattice $\langle 1, i \rangle$. Thus, as the parameter changes, the lattice under consideration remains the same. We first prove the following proposition to show that the families W_μ and \wp_λ are dynamically equivalent.

PROPOSITION 3.2.2. *Let $\lambda \neq 0$ and set $\phi_\lambda(z) = \lambda z$. The map \wp_λ is conformally conjugate under ϕ_λ to W_μ , where $\mu = \gamma^2/\lambda^3$.*

PROOF. Let $\lambda \in \mathbb{C}^*$ and note that

$$(3.5) \quad (\phi_\lambda^{-1} \circ \wp_\lambda \circ \phi_\lambda)(z) = \frac{1}{\lambda} \wp_\lambda(\lambda z).$$

Using the homogeneity equations (1.5) we have that the above is equal to

$$\frac{1}{\lambda^3} \wp_1(z) = \frac{\gamma^2}{\lambda^3} \wp_\gamma(\gamma z),$$

which completes the proof. \square

PROPOSITION 3.2.3. *For every $\mu \in \mathbb{C}^*$ the map $W_{i\mu}$ is conformally conjugate to the map W_μ .*

PROOF. Let $\psi(z) = -iz$. Consider

$$\begin{aligned} (\psi^{-1} \circ W_{i\mu} \circ \psi)(z) &= \frac{1}{-i} W_{i\mu}(-iz) = \frac{1}{-i} i\mu \wp_\gamma(\gamma iz) \\ &= -\mu \wp_\gamma(\gamma iz) = \mu \wp_\gamma(\gamma z) = W_\mu(z). \end{aligned}$$

\square

We define

$$(3.6) \quad N = \mathbb{C}^* / \sim,$$

where $\mu \sim \mu'$ if $\mu' = i\mu$. Using M as defined in Equation (3.3), define the map $h : M \rightarrow N$ by

$$(3.7) \quad h(z) = \frac{\gamma^2}{z^3}.$$

PROPOSITION 3.2.4. *The map $h : M \rightarrow N$ is well defined and one-to-one.*

PROOF. Let $\lambda \sim \lambda'$ and let $[z]_N$ be the equivalence class of z in N (respectively for M). We must show that $[h(\lambda)]_N = [h(\lambda')]_N$. We know that $\lambda' = e^{k\pi i/6}\lambda$ for some $k \in \mathbb{Z}$, so

$$h(e^{k\pi i/6}\lambda) = \frac{\gamma^2}{(e^{k\pi i/6}\lambda)^3} = i^k \frac{\gamma^2}{\lambda^3}.$$

Since

$$\left[i^k \frac{\gamma^2}{\lambda^3} \right]_N = \left[\frac{\gamma^2}{\lambda^3} \right]_N,$$

we have the result. \square

We can now restate the connection between W_μ and \wp_λ .

PROPOSITION 3.2.5. *For $\lambda \in M$ the map $W_{h(\lambda)}$ is conformally conjugate to \wp_λ .*

Since h is an analytic change of coordinates, we have the following result.

PROPOSITION 3.2.6. *The family W_μ parameterized over N forms a reduced holomorphic family.*

Quadratic-Like Behavior.

DEFINITION 3.2.7. *Let U and V be simply connected open subsets of \mathbb{C} , not equal to \mathbb{C} , such that U is relatively compact in V . A map $f : U \rightarrow V$ is quadratic-like if it is a 2-fold branched covering map with one critical point in U .*

For any $z \in \mathbb{C}$ define $\|z\|_1 = |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ and recall that this is a norm on \mathbb{C} considered as a vector space over \mathbb{R} . Let

$$(3.8) \quad c_1^{m,n} = 1/2 + m + ni, \text{ and } c_2^{m,n} = i/2 + m + ni,$$

which are the critical points of W_μ . One useful property of W_μ is that the critical points do not depend on μ . Since $W_\mu(c_3^{m,n}) = 0$ and 0 is a pole, we do not consider it in the context of studying quadratic-like mappings.

Set

$$(3.9) \quad U_i^{m,n} = \{z : \|z - c_i^{m,n}\| < 1/2\}$$

for $i = 1, 2$, which are diamond-shaped regions centered on $c_i^{m,n}$ which tile the plane.

Furthermore, set

$$(3.10) \quad V_\mu = \{\mu z : \operatorname{Re}(z) > 0\}.$$

We identify a fundamental region of N in \mathbb{C}^* . Define

$$N_{quad} = \{z \in \mathbb{C}^* : -\pi/4 \leq \operatorname{Arg}(\mu) \leq \pi/4\} / \sim,$$

where $z \sim z'$ if $z' = i^k z$ for some $k \in \mathbb{Z}$. The following map is a trivial biholomorphism:

$$i : N_{quad} \rightarrow N, \quad i(z) = [z]_N.$$

We defined the “seam” in N_{quad} to be

$$(3.11) \quad N_s = \left\{ [te^{\pi i/4}]_{N_{quad}} : t > 0 \right\}.$$

To keep the notation from being too cumbersome, we define $\hat{N} = N_{quad} \setminus N_s$ which is simply the open wedge $\{z \neq 0 : -\pi/4 < \operatorname{Arg}(z) < \pi/4\}$.

THEOREM 3.2.8. *For $\mu \in \hat{N}$, the following maps are quadratic-like.*

$$m > 0, -m \leq n \leq m, W_\mu : U_1^{m,n} \rightarrow V_\mu,$$

$$m < 0, m \leq n \leq -m - 1, W_\mu : U_2^{m,n} \rightarrow V_{-\mu}.$$

The proof of Theorem 3.2.8 is postponed until later in the chapter so that we can develop the tools needed for the proof.

REMARK 3. *In Figure 3.1 the reader will notice that $U_1^{0,0} \subset V_\mu$ but $\overline{U_1^{0,0}}$ includes the origin, which is not in V_μ . We will address this in the next section as a special case.*

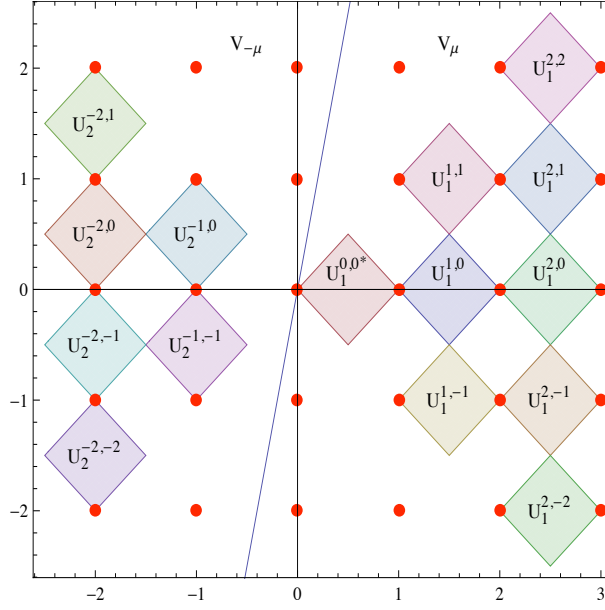


FIGURE 3.1. $U_i^{m,n}$'s as subsets of V_μ or $V_{-\mu}$

Let $M_{quad} = h^{-1}(N_{quad})$, where $h^{-1}(z) = \gamma^{2/3}z^{-1/3}$ and the cube root denotes the principal root $e^{i\theta/3}$. From this we see that

$$M_{quad} = \{z \in \mathbb{C}^* : -\pi/12 \leq \text{Arg}(\mu) \leq \pi/12\} / \sim,$$

where $z \sim z'$ if $z' = e^{k\pi i/6}z$ for some $k \in \mathbb{Z}$. Also define the seam as before: $M_s = h^{-1}(N_s)$.

THEOREM 3.2.9. *For \wp_λ parameterized over $\hat{M} = M_{quad} \setminus M_s$ the following parameterized families of maps are quadratic-like:*

$$m > 0, -m \leq n \leq m, \wp_\lambda : \phi_\lambda(U_1^{m,n}) \rightarrow \phi_\lambda(V_{h(\lambda)}),$$

$$m < 0, m \leq n \leq -m - 1, \wp_\lambda : \phi_\lambda(U_2^{m,n}) \rightarrow \phi_\lambda(V_{-h(\lambda)}),$$

where $\phi_\lambda(z) = \lambda z$.

An *analytic family of quadratic-like maps* is a collection of quadratic-like maps where the domain and range of the maps vary continuously with changes in the parameter

space. The formal definition is given below. We recall that a *proper map* is a continuous map with the property that pre-images of compact sets are compact.

DEFINITION 3.2.10. *Let Λ be a Riemann surface and let $\mathcal{B} = \{f_\lambda : U_\lambda \rightarrow V_\lambda\}$ be a family of quadratic-like mappings. Let*

$$\mathcal{U} = \{(\lambda, z) | \lambda \in \Lambda, z \in U_\lambda\},$$

$$\mathcal{V} = \{(\lambda, z) | \lambda \in \Lambda, z \in V_\lambda\}, \text{ and}$$

$$f(\lambda, z) = (\lambda, f_\lambda(z))$$

The family is analytic if:

- (1) \mathcal{U} and \mathcal{V} are homeomorphic over Λ to $\Lambda \times \mathbb{D}$,
- (2) the projection from the closure of \mathcal{U} in \mathcal{V} to Λ is proper, and
- (3) the map $f : \mathcal{U} \rightarrow \mathcal{V}$ is holomorphic and proper.

For each family described in Theorem 3.2.8 we must show this definition is satisfied.

Again, we note that we treat the $W_\mu : U_1^{0,0} \rightarrow V_\mu$ case separately (Remark 3).

THEOREM 3.2.11. *Let $\Lambda = \hat{N}$. For $m > 0$, $-m \leq n \leq m$, the quadratic-like family*

$$\{W_\mu : U_1^{m,n} \rightarrow V_\mu\}$$

is analytic; and for $m < 0$, $m \leq n \leq -m - 1$, the quadratic-like family

$$\{W_\mu : U_2^{m,n} \rightarrow V_{-\mu}\}$$

is analytic.

Stating the result in terms of \wp_λ , we have the following theorem:

THEOREM 3.2.12. Let $\Lambda' = \hat{M}$. For $m > 0$, $-m \leq n \leq m$, the quadratic-like family

$$\{\phi_\lambda : \phi_\lambda(U_1^{m,n}) \rightarrow \phi_\lambda(V_{h(\lambda)})\}$$

is analytic; and for $m < 0$, $m \leq n \leq -m - 1$, the quadratic-like family

$$\{\phi_\lambda : \phi_\lambda(U_2^{m,n}) \rightarrow \phi_\lambda(V_{-h(\lambda)})\}$$

is analytic.

The canonical example of an analytic family of quadratic-like mappings are the maps $p_c(z) = z^2 + c$. However, one cannot use $p_c : \mathbb{C} \rightarrow \mathbb{C}$, as \mathbb{C} is not conformally equivalent to the open unit disk. One must make a choice of U_c and V_c for each value of c . At first glance, this freedom of choice may make some definitions ill-defined. For instance, the filled Julia set for p_c ,

$$(3.12) \quad K(p_c) = \{z : p_c^n(z) \text{ is bounded}\},$$

is also defined in terms of the choice of U_c and V_c . For example, see Definition 3.2.14. Thus, to avoid misleading the reader we specify one possible choice of U_c and V_c . Furthermore, the techniques used in this elementary case will be used later.

PROPOSITION 3.2.13. Let $p_c(z) = z^2 + c$. Let $U_c = \{z : |z| < 3\}$ and $V_c = p_c(U_c) = \{z : |z - c| < 9\}$. For $\Lambda = \{|c| < 4\}$, the map $p_c : U_c \rightarrow V_c$ is an analytic family of quadratic-like mappings.

PROOF. First we note that $p_c : U_c \rightarrow V_c$ is a 2-1 covering with one critical point in U_c , namely 0, for each c . We must verify that $\overline{U_c} \subset V_c$. We have that $V_c = p_c(U_c)$ which is a ball of radius $3^2 = 9$ centered at c . If $|c| < 4$, a simple triangle inequality argument

gives that $\overline{U_c} \subset V_c$. In fact, V_c always contains a ball of radius 5 centered at the origin.

We now need to check the analyticity of the family.

(1) We can state explicit homeomorphisms. Define the map $(z, c) \mapsto (z/(2 + \epsilon), c)$.

This is a homeomorphism of \mathcal{U} to $\mathbb{D} \times \Lambda$. For the second map define $(z, c) \mapsto ((z - c)/(2 + \epsilon)^2, c)$. This is again a homeomorphism of \mathcal{V} to $\mathbb{D} \times \Lambda$.

(2) Since $\overline{\mathcal{U}}$ is compact, this is trivial.

(3) Properness is again simple, because the inverse image of a closed and bounded set is closed and bounded in U_c . Since p_c is analytic in both its argument and parameter we have the result.

□

These choices of U_c and V_c are somewhat arbitrary. They are chosen essentially to have the property $\overline{U_c} \subset p_c(U_c) = V_c$. It is not yet clear that given different choices of U_c and V_c that there is any equivalence between the families. This question was answered by the Straightening Theorem of Douady and Hubbard in [25]. This is the subject of the next section.

We define the filled Julia set of a quadratic-like map. We now assume that $f_\lambda : U_\lambda \rightarrow V_\lambda$ is an analytic family quadratic-like mappings.

DEFINITION 3.2.14. *We define the filled Julia set of $f_\lambda : U_\lambda \rightarrow V_\lambda$ of an analytic family of quadratic-like mappings as*

$$K_{f_\lambda} = K_\lambda = \bigcap_{n \geq 0} f_\lambda^{-n}(U_\lambda) = \{z \in U_\lambda \mid f_\lambda^n(z) \in U_\lambda, \text{ for all } n \geq 0\}.$$

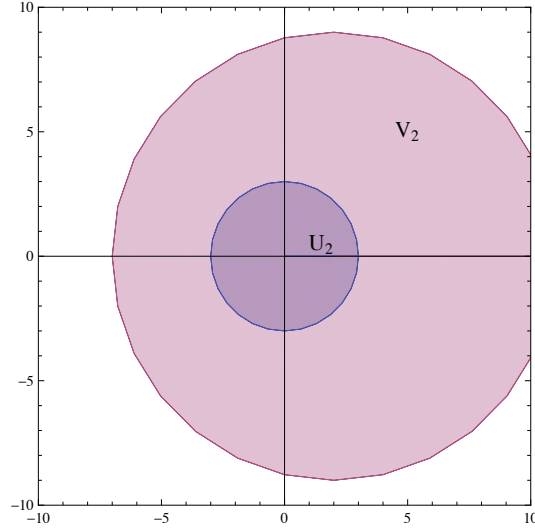


FIGURE 3.2. $p_2 : U_2 \rightarrow V_2$ and $\overline{U_2} \subset V_2$

DEFINITION 3.2.15. We define the Mandelbrot-like set $M_{\mathcal{B}}$ of the analytic family \mathcal{B} , as defined in Definition 3.2.10, to be

$$M_{\mathcal{B}} = \{\lambda \mid K_{\lambda} \text{ is connected}\}.$$

The standard Mandelbrot set or just *Mandelbrot set* is:

$$\mathcal{M} = \{c \mid K_{p_c} \text{ is connected}\},$$

where $p_c = z^2 + c$ as above. It is pictured in Figure 3.3.

We show that the sets $\mathcal{C}_i^{m,n}$ and $\mathcal{D}_i^{m,n}$, defined below, are indeed $M_{\mathcal{B}}$, where $\mathcal{B} = \{W_{\mu} : U_i^{m,n} \rightarrow W_{\mu}(U_i^{m,n}) : \mu \in \hat{M}\}$.

DEFINITION 3.2.16. For $i = 1, 2$, $(m, n) \in \mathbb{Z}^2$, and $\mu \in \hat{M}$, we define

$$(3.13) \quad \mathcal{C}_i^{m,n} = \{\mu : W_{\mu}^k(c_i^{m,n}) \in U_i^{m,n} \text{ for all } k \geq 0\},$$

where $c_i^{m,n}$ are defined in equation (3.8).

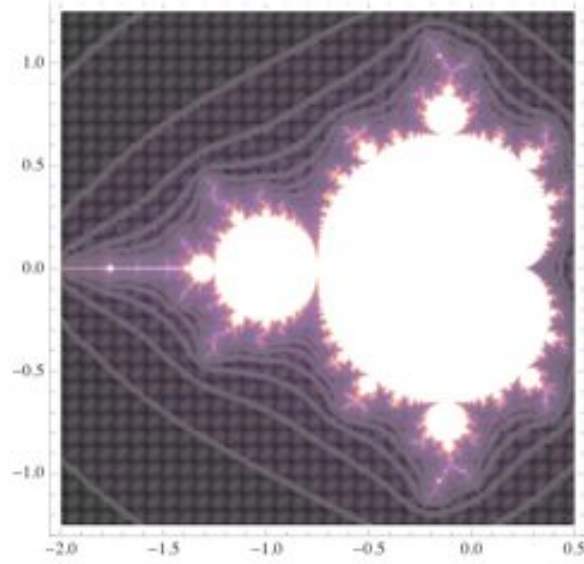


FIGURE 3.3. The Mandelbrot Set

Connecting back to \wp_λ , we formulate the equivalent definition.

DEFINITION 3.2.17. *For $i = 1, 2$, $(m, n) \in \mathbb{Z}^2$, and $\lambda \in \hat{N}$, we define*

$$\mathcal{D}_i^{m,n} = \{\lambda : \wp_\lambda^k(\phi_\lambda(c_i^{m,n})) \in \phi_\lambda(U_i^{m,n}) \text{ for all } k \geq 0\}.$$

PROPOSITION 3.2.18. *For each $i = 1, 2$ and $(m, n) \in \mathbb{Z}^2$ we have the following map between $\mathcal{D}_i^{m,n}$ and $\mathcal{C}_i^{m,n}$:*

$$h(\mathcal{D}_i^{m,n}) = \mathcal{C}_i^{m,n},$$

where h is defined in equation (3.7).

Our main result of this chapter is the following. Again, we postpone the proof until later in the chapter.

THEOREM 3.2.19. *For $i = 1, 2$ and all $(m, n) \in \mathbb{Z}^2$, the set $\mathcal{C}_i^{m,n}$ is homeomorphic to the standard Mandelbrot set \mathcal{M} . Furthermore this homeomorphism is analytic on the interior of $\mathcal{C}_i^{m,n}$.*

Connectedness of the Filled Julia Set.

The following result is a key tool in identifying if a parameter value c lies in \mathcal{M} for the family $p_c(z) = z^2 + c$. Furthermore, using the Straightening Theorem 3.3.3, this result is fundamental in locating the Mandelbrot-like sets for any analytic family of quadratic like mappings.

PROPOSITION 3.2.20. [25] *Let $f : U \rightarrow V$ be quadratic-like. The filled Julia set, K_f , is connected if and only if the critical point is in K_f .*

We present a partial proof of this for the map $p_c(z) = z^2 + c$. The general result for quadratic-like mappings follows from the Straightening Theorem presented in the next section.

PROOF. This proof can be found in [41]. Suppose that the critical point 0 lies in the filled Julia set. Denote the basin of infinity by $B_c = \mathbb{C}_\infty \setminus K_{p_c}$. Since infinity is a super-attracting fixed point, we have a Böttcher coordinate ϕ , which extends to a map $\phi : B_c \rightarrow \mathbb{D}$ since B_c contains only the critical point at infinity ([41, Theorem 9.3]). This naturally gives rise to a map

$$\hat{\phi} : \mathbb{C} \setminus K_{p_c} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}},$$

where $\hat{\phi} = 1/\phi$. Consider the annuli $A_\epsilon = \{z : 1 < |z| < 1 + \epsilon\}$. Now $\hat{\phi}^{-1}(A_\epsilon)$ is a connected set with compact closure that must contain the Julia set, which is ∂B_c . Thus,

$$J(p_c) = \bigcap_{\epsilon > 0} \overline{\hat{\phi}^{-1}(A_\epsilon)}$$

is connected, and so K_{p_c} is connected.

To prove that K_{p_c} is disconnected if $0 \in B_c$, we refer to [41].

□

To determine the location of \mathcal{M} we prove the following.

PROPOSITION 3.2.21. *The Mandelbrot set \mathcal{M} is contained in a closed ball of radius 3 centered at zero.*

PROOF. Suppose that $|c| > 3$. Then the critical value, $p_c(0) = c \notin U_c = \{z : |z| < 3\}$ and so $c \notin K_{p_c}$, which implies that $c \notin \mathcal{M}$. Thus, $\mathcal{M} \subset \{|z| \leq 3\}$. □

Therefore, in general one looks for parameter values so that the critical value of the corresponding map lies in the domain on which a map is quadratic-like; i.e. if c_λ is the critical point of $f_\lambda : U_\lambda \rightarrow V_\lambda$, one solves for λ so that $f_\lambda(c_\lambda) \in U_\lambda$. This provides a region, $A \subset \Lambda$, which contains $M_{\mathcal{B}}$.

3.3. The Straightening Theorem

We now introduce the tools to connect the dynamics of Weierstrass elliptic functions on square lattices to the classical quadratic map $p_c(z) = z^2 + c$. We begin with the formal definition of a quasi-conformal mapping.

DEFINITION 3.3.1. *Let U be an open subset of \mathbb{C} . Let $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$. Let $\phi(x, y) = \phi_1(x, y) + i\phi_2(x, y)$ where $x + iy$ is identified as (x, y) and suppose that ϕ has continuous partial derivatives on U . A map $\phi : U \rightarrow \mathbb{C}$ is quasi-conformal if*

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z},$$

where μ is a Lebesgue measurable function with $|\mu| < 1$.

One can think of quasi-conformal maps as maps that take small circles to small ellipses. If ϕ is conformal, then $\mu(z) = 0$, so one can think of $\mu(z)$ as a measure of distortion from being conformal.

DEFINITION 3.3.2. *Two quadratic-like maps $f : U \rightarrow V$ and $g : U' \rightarrow V'$ are hybrid equivalent if there is a quasi-conformal homeomorphism ϕ mapping a neighborhood W of K_f to a neighborhood W' of K_g so that $g \circ \phi = \phi \circ f$ and ϕ is conformal on K_f .*

THEOREM 3.3.3. **[25, The Straightening Theorem]** *Let $f : U \rightarrow V$ be a quadratic-like map. The map f is hybrid equivalent to $p_c : U_c \rightarrow V_c$ for some c . If K_f is connected, the conjugacy is unique.*

This theorem gives rise to a map $\chi : M_{\mathcal{B}} \rightarrow \mathcal{M}$, $\chi(\lambda) = c$, where \mathcal{M} is the Mandelbrot set. The following theorem gives a sufficient condition for the map χ to be a homeomorphism.

THEOREM 3.3.4. **[25]** *Let $A \subset \Lambda$ be a closed set of parameters homeomorphic to a disk and containing $M_{\mathcal{B}}$. Let ω_λ be the critical point of f_λ , and suppose that for each $\lambda \in \Lambda \setminus A$ the critical value $f_\lambda(\omega_\lambda) \in V_\lambda \setminus U_\lambda$. Assume also that as λ goes once around ∂A , the vector $f_\lambda(\omega_\lambda) - \omega_\lambda$ turns once around 0. Then the map $\chi : M_{\mathcal{B}} \rightarrow \mathcal{M}$ is a homeomorphism and is analytic on the interior of $M_{\mathcal{B}}$.*

3.4. The Proofs of Theorems 3.2.8 and 3.2.11

In this section we develop the technical material to prove the main theorems of the chapter. We restate them now.

Theorem 3.2.8 For $\mu \in \hat{N}$, the following maps are quadratic-like:

$$m > 0, -m \leq n \leq m, W_\mu : U_1^{m,n} \rightarrow V_\mu,$$

$$m < 0, m \leq n \leq -m - 1, W_\mu : U_2^{m,n} \rightarrow V_{-\mu}.$$

Theorem 3.2.11 Let $\Lambda = \hat{N}$. For $m > 0, -m \leq n \leq m$, the quadratic-like family

$$\{W_\mu : U_1^{m,n} \rightarrow V_\mu\}$$

is analytic; and for $m < 0, m \leq n \leq -m - 1$, the quadratic-like family

$$\{W_\mu : U_2^{m,n} \rightarrow V_{-\mu}\}$$

is analytic.

Assume that $\mu \in \hat{N}$. We focus our attention on W_1 since $W_\mu = \mu W_1$. Thus, if $W_1 : A \rightarrow B$, then $W_\mu : A \rightarrow \mu B$.

Recall that for $i = 1, 2$

$$U_i^{m,n} = \{z : \|z - c_i^{m,n}\|_1 < 1/2\},$$

which can be described as the interior of a diamond centered at $c_i^{m,n}$ with vertices $c_i^{m,n} + 1/2, c_i^{m,n} - 1/2, c_i^{m,n} + i/2$, and $c_i^{m,n} - i/2$.

As a consequence of Theorem 1.2.7 we have the following.

PROPOSITION 3.4.1. *The map W_μ is of order 2, i.e., 2-1 on a fundamental region.*

The reflection of a point $z_0 + z$ through the point z_0 is $z_0 - z$. We say a map S is *symmetric about a point* $z_0 \in \mathbb{C}$ if $S(z_0 + z) = S(z_0 - z)$.

PROPOSITION 3.4.2. *The function W_μ is symmetric about $c_i^{m,n}$, i.e., $W_\mu(c_i^{m,n} + z) = W_\mu(c_i^{m,n} - z)$.*

PROOF. The point $2c_i^{m,n}$ is a lattice point and W_μ is even, so

$$W_\mu(c_i^{m,n} - z) = W_\mu(z - c_i^{m,n}) = W_\mu(z - c_i^{m,n} + 2c_i^{m,n}) = W_\mu(c_i^{m,n} + z).$$

□

LEMMA 3.4.3. *The map $W_1 : \partial U_1^{m,n} \rightarrow i\mathbb{R} \cup \{\infty\}$ is 2-1 and onto.*

PROOF. We note that $W_1(c_1^{m,n} + 1/2) = \infty = W_1(c_1^{m,n} - 1/2)$, $W_1(c_1^{m,n} + i/2) = 0 = W_1(c_1^{m,n} - i/2)$, and W_1 has no critical points on the lines connecting the vertices. We now show that the image of the lines connecting the vertices is the imaginary axis. First note that the points on the edges are $\langle 1, i \rangle$ equivalent to points on $e^{i\pi/4}t$ or $e^{-i\pi/4}t$ for some $-\sqrt{2}/2 < t < \sqrt{2}/2$ ($t \neq 0$). Computing,

$$W_1(e^{i\pi/4}t) = \wp_\gamma(\gamma e^{i\pi/4}t) = (e^{i\pi/4})^{-2} \wp_{e^{-i\pi/4}\gamma}(t) = -i \wp_{e^{-i\pi/4}\gamma}(t)$$

and

$$W_1(e^{-i\pi/4}t) = \wp_\gamma(\gamma e^{-i\pi/4}t) = (e^{-i\pi/4})^{-2} \wp_{e^{i\pi/4}\gamma}(t) = i \wp_{e^{i\pi/4}\gamma}(t).$$

In both cases, $\wp_{e^{-i\pi/4}\gamma}$ and $\wp_{e^{i\pi/4}\gamma}$, the lattice is real ($\overline{\Omega} = \Omega$) and so $\wp_{e^{-i\pi/4}\gamma}(t)$ and $\wp_{e^{i\pi/4}\gamma}(t)$ are both real. From this we obtain the result. □

THEOREM 3.4.4. *The map*

$$W_1 : U_1^{m,n} \rightarrow V_1$$

is a 2-1 branched covering map.

PROOF. We know that the image of $U_1^{m,n}$ is connected, since W_1 is continuous. We also have $W_1(c_1^{m,n}) = 1$, from equation (1.15). Suppose that $\Re(W_1(z)) \leq 0$. Consider a curve α connecting $c_1^{m,n}$ to z . Then $W_1(\alpha)$ must intersect the imaginary axis. This is a contradiction, since W_1 is an elliptic function of order 2, $U_1^{m,n}$ is contained in a period parallelogram,

$$\{1/2 + m + ni + s + ti : -1/2 \leq s, t < 1/2\},$$

and Lemma 3.4.3 gives that $W_1 : \partial U_1^{m,n} \rightarrow i\mathbb{R} \cup \{\infty\}$ is 2-1 and onto. Thus, $W_1 : U_1^{m,n} \rightarrow V_1$. Since $U_1^{m,n}$ is symmetric about $c_1^{m,n}$, Proposition 3.4.2 gives that the map is 2-1. Lastly, we need to show that the map is onto.

Using a similar argument one can show that

$$W_1 : U_2^{m,n} \rightarrow V_{-1}.$$

Since $\overline{U_1^{m,n} \cup U_2^{m,n}}$ is the closure of a fundamental region,

$$W_1 : \overline{U_1^{m,n} \cup U_2^{m,n}} \rightarrow \overline{V_1 \cup V_{-1}}$$

must be onto, by Theorem 1.2.7. Thus, decomposing, we have

$$W_1 : U_1^{m,n} \rightarrow V_1$$

$$W_1 : U_2^{m,n} \rightarrow V_{-1}$$

$$W_1 : \partial U_1^{m,n} \cup \partial U_2^{m,n} \rightarrow i\mathbb{R},$$

which implies that $W_1 : U_1^{m,n} \rightarrow V_1$ is onto and thus a 2-1 branched cover. □

COROLLARY 3.4.5. *The map*

$$W_1 : U_2^{m,n} \rightarrow V_{-1}$$

is a 2-1 branched covering map. Furthermore, we have

$$W_\mu : U_1^{m,n} \rightarrow V_\mu$$

and

$$W_\mu : U_2^{m,n} \rightarrow V_{-\mu}$$

are 2-1 branched covering maps.

We justify the restrictions on m, n . These restrictions arise due to our choice of fundamental region of N , N_{quad} .

PROPOSITION 3.4.6. *For $|Arg(\mu)| < \pi/4$, the set $\overline{U_1^{m,n}} \subset V_\mu$ if and only if $m > 0$ and $-m \leq n \leq m$. Furthermore, for $|Arg(\mu)| < \pi/4$, the set $\overline{U_2^{m,n}} \subset V_{-\mu}$ if and only if $m < 0$ and $m \leq n \leq -m - 1$.*

PROOF. We have

$$\bigcap_{|Arg(\mu)| < \pi/4} V_\mu = \{z : |Arg(z)| < \pi/4\}$$

which implies $\overline{U_1^{m,n}} \subset V_\mu$ if and only if $m > 0$ and $-m \leq n \leq m$. The proof of the second part is similar. □

PROOF. (Proof of Theorem 3.2.8) Theorem 3.2.8 follows immediately from Proposition 3.4.6 and Corollary 3.4.5. □

We now prove that the families are analytic. These ideas are almost identical to the ones presented in the proof of Proposition 3.2.13.

PROOF. (Proof of Theorem 3.2.11) Fix i, m, n satisfying the hypotheses of Theorem 3.2.8. We proved that the maps are quadratic-like in Theorem 3.2.8. It remains to show

that the family $\{W_\mu : U_i^{m,n} \rightarrow V_{\pm\mu}\}$ satisfies the three conditions required for analyticity in Definition 3.2.10.

- (1) We show that $\mathcal{U} = \{(\mu, z) : \mu \in \hat{N}, z \in U_i^{m,n}\}$ is homeomorphic to $\hat{N} \times \mathbb{D}$. We have that $U_i^{m,n}$ is homeomorphic to an open disk by the Riemann Mapping Theorem. Call this homeomorphism $h_1 : U_i^{m,n} \rightarrow \mathbb{D}$. The desired homeomorphism of \mathcal{U} to $\hat{N} \times \mathbb{D}$ is $(\mu, z) \mapsto ((\mu, h_1(z)))$. The case for $\mathcal{V} = \{(u, z) : \mu \in \hat{N}, z \in V_\mu\}$ is almost identical. Very simply, $V_\mu = \mu V_1$. Since V_1 is homeomorphic to the open unit disk by the Riemann Mapping Theorem, we can repeat the process. Let $h_2 : V_1 \rightarrow \mathbb{D}$ now be the conformal homeomorphism of V_1 to the open unit disk. Let $h_3(\mu, z) = \mu z$. The desired homeomorphism of \mathcal{V} to $\hat{N} \times \mathbb{D}$ is $(\mu, z) \mapsto (\mu, h_2(z/\mu)) = (\mu, (h_2 \circ h_3^{-1})(z))$. (The argument is similar for $-\mu$ when $i = 2$.)
- (2) We must show the properness of $(\mu, z) \mapsto \mu$ where $z \in \overline{U_i^{m,n}} \subset V_\mu$. The inverse image of a compact set K is closed and contained in $K \times \overline{U_i^{m,n}}$, and so is bounded, thus compact.
- (3) To show that $(\mu, z) \mapsto (\mu, W_\mu(z))$ for $(\mu, z) \in \mathcal{U}$ is proper, we have only to observe that since $U_i^{m,n}$ is a bounded set, the inverse image of a compact set K is closed and bounded in $K \times U_i^{m,n}$ and thus compact. By viewing the definition of the map $W_\mu(z) = \mu \wp_\gamma(\gamma z)$, we can clearly see that the map is analytic in the parameter. Since $U_i^{m,n}$ contains no poles, W_μ is analytic in z . Thus, the map $(\mu, z) \mapsto (\mu, W_\mu(z))$ for $(\mu, z) \in \mathcal{U}$ is holomorphic.

□

3.5. The Case of $U_1^{0,0}$ Tangent to the Origin

The obstacle in showing the family $W_\mu : U_1^{0,0} \rightarrow V_\mu$ is quadratic-like is that $\overline{U_1^{0,0}}$ contains the origin which is not contained in V_μ for $\mu \in \hat{N}$. Our strategy to rectify the problem is to remove a small ball around the origin, which then forces one to remove the symmetric (with respect to the critical point $1/2$) region about the other corner at 1. After doing this, the image of the modified region under W_μ is no longer the full half-plane. We will also need to modify the parameter space \hat{N} slightly.

DEFINITION 3.5.1. *Let $r > 0$. We define*

$$(3.14) \quad U_1^{0,0}(r) = U_1^{0,0} \setminus (B_r(0) \cup B_r(1)),$$

where $B_r(z_0) = \{z : |z - z_0| < r\}$.

PROPOSITION 3.5.2. *If $r < 1/\gamma$ and $z \in B_r(0)$ then*

$$|W_1(z)| > \frac{1}{\gamma^2} \left(\left| \frac{1}{z^2} \right| - \gamma^2 \frac{3}{14} \right).$$

PROOF. We prove this using the Laurent series of $W_1(z) = 1/\gamma^2 \wp_1(z) = \wp_\gamma(\gamma z)$. We recall from 1.2.9 that $g_2 = 4e_1^2 = 4$, since $\lambda = \gamma$. The Laurent series centered at zero for $W_1(z)$ is $[\mathbf{1}]$:

$$(3.15) \quad \frac{1}{(\gamma z)^2} + \sum_{k=2}^{\infty} c_k (\gamma z)^{2k-2},$$

where $c_2 = g_2/20 = 1/5$, $c_3 = g_3/28 = 0$, and

$$c_k = \frac{1}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}$$

for $k > 3$. As a consequence of $c_3 = 0$ we have that $c_k = 0$, when k is odd. Furthermore, we have estimates on $|c_{2l}| < c_2 \left(\frac{c_2}{3}\right)^{l-1}$ ([1]). So we have that

$$|c_{2l}| < \frac{1}{5} \left(\frac{1}{15}\right)^{l-1}.$$

Examining the polynomial terms and using geometric series we have:

$$(3.16) \quad \left| \sum_{l=1}^{\infty} c_{2l}(\gamma z)^{4l-2} \right| \leq \frac{|(\gamma z)^2|}{5} \sum_{l=1}^{\infty} \left| \frac{(\gamma z)^4}{15} \right|^{l-1} = \frac{|(\gamma z)^2|}{5} \frac{1}{1 - \left| \frac{(\gamma z)^4}{15} \right|}.$$

Thus using the triangle inequality,

$$(3.17) \quad \left| \frac{1}{(\gamma z)^2} + \sum_{l=1}^{\infty} c_{2l}(\gamma z)^{4l-2} \right| \geq \left| \left| \frac{1}{(\gamma z)^2} \right| - \left| \frac{|(\gamma z)^2|}{5} \frac{1}{1 - \left| \frac{(\gamma z)^4}{15} \right|} \right| \right|;$$

and if $|z| < 1/\gamma$, we have

$$(3.18) \quad \left| \frac{1}{(\gamma z)^2} + \sum_{l=1}^{\infty} c_{2l}(\gamma z)^{4l-2} \right| \geq \left| \left| \frac{1}{(\gamma z)^2} \right| - \frac{3}{14} \right|,$$

giving the result after factoring out $1/\gamma^2$. □

Using the same techniques used to prove Corollary 3.4.4, we have the following.

THEOREM 3.5.3. *Let $r < 1/100$. For $\mu \in \hat{N}$ with $|\mu| > r/2 > 0$, we have that*

$$W_{\mu} : U_1^{0,0}(r) \rightarrow W_{\mu}(U_1^{0,0}(r))$$

is a quadratic-like map.

PROOF. It has already been established that $W_{\mu} : U_1^{0,0}(r) \rightarrow W_{\mu}(U_1^{0,0}(r))$ is a 2-1 cover with one critical point. What is left to show is that $\overline{U_1^{0,0}(r)} \subset W_{\mu}(U_1^{0,0}(r))$. It is sufficient to show that $|W_{\mu}(z)| > 1$ for $z \in B_r(0)$. From Proposition 3.5.2 we have

$$|W_{\mu}(z)| \geq |\mu| \left(\left| \frac{1}{z^2} \right| - \frac{3\gamma^2}{14} \right) > \frac{1}{100\gamma^2} \left(10000 - \frac{3\gamma^2}{14} \right) > 10.$$

□

Analyticity of the quadratic-like family $W_\mu : U_1^{0,0}(r) \rightarrow W_\mu(U_1^{0,0}(r))$ follows using the proof in Theorem 3.2.11 by replacing $U_i^{m,n}$ with $U_1^{0,0}(r)$ and V_μ with $W_\mu(U_1^{0,0}(r))$.

THEOREM 3.5.4. *Let $r < 1/100$. The family*

$$W_\mu : U_1^{0,0}(r) \rightarrow W_\mu(U_1^{0,0}(r)),$$

parameterized over $\mu \in \hat{N}$ with $|\mu| > r/2 > 0$, is an analytic family of quadratic-like maps.

Restating Theorem 3.5.4 in terms of \wp_λ , we have the following.

COROLLARY 3.5.5. *Let $r < 1/100$. The family*

$$\wp_\lambda : \phi_\lambda(U_1^{0,0}(r)) \rightarrow \wp_\lambda(\phi_\lambda((U_1^{0,0}(r))))$$

parameterized over $\lambda \in \hat{M}$ with $|\frac{\gamma^2}{\lambda^3}| > r/2 > 0$, is an analytic family of quadratic-like maps.

We note that we can make r as small as we like. Thus we can consider $W_\mu : U_1^{0,0} \rightarrow V_\mu$ to be an analytic family of quadratic-like maps, with the understanding that one must remove small neighborhoods about 0 and 1.

The Dynamics Along the Seam.

All the theorems up to this point concerning W_μ have been about $\mu \in \hat{N}$ which we recall is N_{quad} , except the “seam” N_s , where $\arg \mu = \pm\pi/4$. The dynamics along N_s in N_{quad} are treated in [30]. We present the theory below.

THEOREM 3.5.6. [30, Proposition 1.1] *If a lattice Ω is rhombic, then the Julia set $J(\wp_\Omega)$ is connected if and only if it is the entire Riemann sphere \mathbb{C}_∞ .*

We have shown that in the setting for square lattices the Julia set is always connected. We claim first that the seam of M_{quad} as mentioned before is the set of parameters which produce “rhombic” dynamics.

PROPOSITION 3.5.7. *If $\arg \lambda = \pm\pi/12$, then \wp_λ is conformally conjugate to \wp'_λ , where $\arg \lambda' = \mp\pi/4$ and so $\overline{\lambda'} = i\lambda'$.*

PROOF. By Proposition 3.1.4, we have that \wp_λ is conformally conjugate to $\wp_{e^{k\pi i/6}\lambda}$ for any $k \in \mathbb{Z}$. Assume that $\arg \lambda = \pi/12$. Let $k = -2$ and set $\lambda' = e^{-2\pi i/6}\lambda$. Thus, $\arg(\lambda') = -\pi/4$. Since $i\lambda' = \overline{\lambda'}$, we are done. The case where $\arg \lambda = -\pi/12$ is similar. \square

Since $h : M_{quad} \rightarrow N_{quad}$ preserves the seam, i.e. $h : M_s \rightarrow N_s$, we obtain similar results for $W_{h(\lambda)}$ where $\lambda \in M_s$.

COROLLARY 3.5.8. *Let $\lambda \in M_{quad}$. If $\arg(\lambda) = \pm\pi/12$, then $J(\wp_\lambda) = \mathbb{C}_\infty = J(W_{h(\lambda)})$. Consequently, if $\arg \mu = \pm\pi/4$ ($\mu \in N_s$) then $J(W_\mu) = \mathbb{C}_\infty$.*

3.6. Mandelbrot Sets in the Parameter Space, the Proof of Theorem 3.2.19

In this section we prove that the parameter space contains copies of the Mandelbrot set. In addition, this will imply that there are parameters λ such that \wp_λ has Siegel disks. This is proved in Corollary 3.7.7. We restate the theorem that we now prove in this section. Before concluding the section with the proof, we develop the notation needed to describe the locations of the Mandelbrot sets.

Theorem 3.2.19 *For all $i = 1, 2$ and all $(m, n) \in \mathbb{Z}^2$, the set $\mathcal{C}_i^{m,n}$ is homeomorphic to the standard Mandelbrot set \mathcal{M} . Furthermore this homeomorphism is analytic on the interior of $\mathcal{C}_i^{m,n}$.*

To find regions in parameter space where the filled Julia set is connected, we are interested in when the critical point of a quadratic-like map is actually in the filled Julia set.

We use the following notation for the filled Julia set:

$$K_i^{m,n}(\mu) = \bigcap_{k \geq 0} W_\mu^{-k}(U_i^{m,n}),$$

where $i=1,2$. Recall that the 1-norm is defined as $\|z\|_1 = |Re(z)| + |Im(z)|$ where \mathbb{C} is thought of as a vector space over \mathbb{R} . The following is a statement that follows directly from the definition of $K_i^{m,n}(\mu)$ ($i=1,2$).

THEOREM 3.6.1. *The orbit of $c_i^{m,n}$ lies in the filled Julia set $K_i^{m,n}(\mu)$ ($i=1,2$) if and only if*

$$\|W_\mu^k(c_i^{m,n}) - c_i^{m,n}\|_1 < \frac{1}{2}$$

for all $k \geq 0$. This is equivalent to $\mu \in \mathcal{C}_i^{m,n}$.

To use Theorem 3.3.4 we need to define the regions in parameter space, \hat{N} , where the Mandelbrot sets occur. We define these to be

$$A_1^{m,n} = \overline{U_1^{m,n}}$$

and

$$A_2^{m,n} = \overline{-U_2^{m,n}} = \overline{U_2^{-m,-n-1}}$$

where the closure is in \hat{N} .

REMARK 4. The indexing and locations of $\mathcal{C}_i^{m,n}$ and $A_i^{m,n}$ are the same in the sense that $\mathcal{C}_i^{m,n} \subset A_i^{m,n}$. This inclusion is in the proof of Theorem 3.2.19 at the end of the section.

Taking the union over the allowable choices of m, n so that $\mu \in \overline{U_1^{m,n}}$ and $\mu \in \hat{N}$ or $-\mu \in \overline{U_2^{m,n}}$ and $\mu \in \hat{N}$ we have the following theorem.

THEOREM 3.6.2. The parameter space N_{quad} is the following union:

$$N_{quad} = \left(\bigcup_{\substack{i=1, m \geq 0, -m \leq n \leq m \\ i=2, m < 0, m \leq n \leq -m-1}} A_i^{m,n} \right) \cup N_s$$

PROOF. This theorem is best illustrated by Figures 3.4 and 3.5. Considering the dynamic plane on the left in Figure 3.5, one can see that exactly one of the following must occur for $\mu \in N_{quad}$:

- (1) $\mu \in \overline{U_1^{m,n}}$ for $m \geq 0, -m \leq n \leq m$,
- (2) $-\mu \in \overline{U_2^{m,n}}$ for $m < 0, m \leq n \leq -m - 1$, or
- (3) $\mu \in N_s$ (and so also $-\mu \in -N_s$).

□

The restrictions on m, n can be relaxed by using the preceding theorem. One needs only to recall that W_μ is conjugate to $W_{i\mu}$ by Proposition 3.2.3. Thus, covering the plane with $i^k N_{quad}$ where $k = 0, 1, 2, 3$ one no longer needs to restrict the allowable choices of m, n .

REMARK 5. If one relaxes the restriction on m, n, i by allowing $\mu \in \mathbb{C}^*$ one loses the unique association between \wp_λ and W_μ . However, the relaxation only produces the

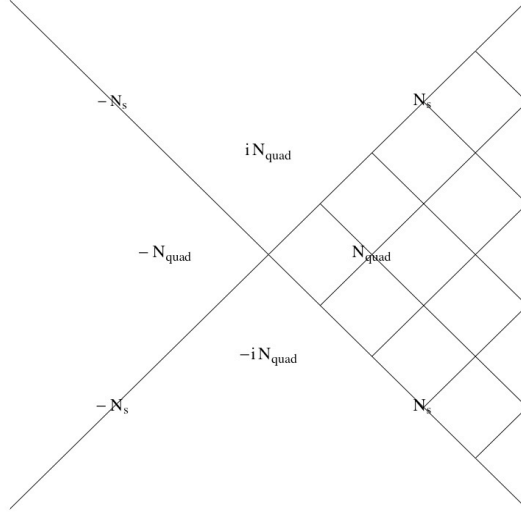


FIGURE 3.4. The parameter μ -plane for W_μ

following problem:

$$h : \mathbb{C}^* \rightarrow \mathbb{C}^*$$

is a three-to-one covering map. One may consider branches of h^{-1} to relate results back to \wp_λ . Taking a branch cut along the negative real axis, we let $h^{-1}\mathbb{C}^* = X_1 \cup X_2 \cup X_3$ where $X_1 = \{z : -\pi/3 \leq \arg z \leq \pi/3\}/\sim$, $X_2 = \{z : \pi/3 \leq \arg z \leq \pi\}/\sim$, and $X_3 = \{z : -\pi \leq \arg z \leq -\pi/3\}/\sim$ where $z \sim z'$ if $e^{2k\pi/3}z = z'$ for some $k \in \mathbb{Z}$. Since

$$h : X_i \rightarrow \mathbb{C}^*$$

is a biholomorphism, one may relax the restrictions on (m, n, i) induced by the coordinates of M_{quad} if one allows $\lambda \in X_i$ for only one of $i = 1, 2$, or 3 .

Relating back to \wp_λ , we recall that \wp_λ is conformally conjugate to $W_{h(\lambda)}$, where $h(\lambda) = \gamma^2/\lambda^3$. Thus, the regions in M_{quad} corresponding to the $A_i^{m,n}$'s in N_{quad} are $h^{-1}(A_i^{m,n})$. It is easily seen that the centers of the Mandelbrot sets $h^{-1}(\mathcal{C}_i^{m,n})$, $\lambda = (\gamma^2/(1/2+m+ni))^{1/3}$ and $\lambda = (\gamma^2/(i/2+m+ni))^{1/3}$, cluster on the origin; and there is a largest center in

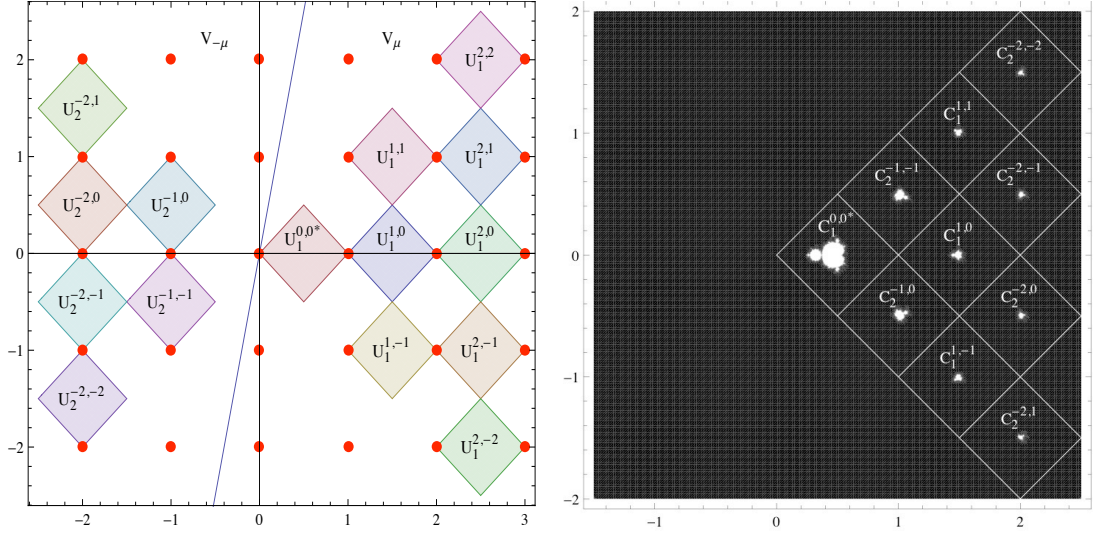


FIGURE 3.5. The locations of $C_i^{m,n}$

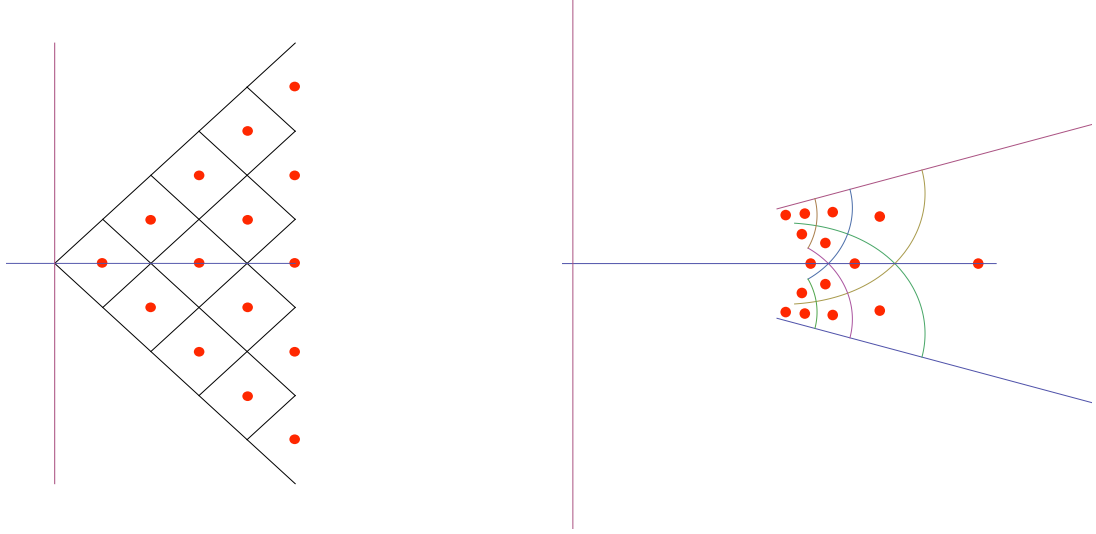


FIGURE 3.6. Centers of Mandelbrot set in \hat{N} and \hat{M} respectively

M_{quad} , namely $\lambda = (\gamma^2/(1/2))^{1/3} = (2\gamma^2)^{1/3}$. We illustrate this in Figure 3.6. One can see the difficulties in describing the regions containing Mandelbrot sets in λ -space without utilizing the family W_μ .

PROOF. (Proof of Theorem 3.2.19) It is enough to verify that the conditions of Theorem 3.3.4 are all satisfied. We recall that

$$\mathcal{C}_1^{m,n} = \left\{ \mu : \|W_\mu^k(c_1^{m,n}) - c_1^{m,n}\|_1 < \frac{1}{2} \text{ for all } k \geq 0 \right\} \subset \overline{\left\{ \mu : \|\mu - c_1^{m,n}\|_1 < \frac{1}{2} \right\}} = A_1^{m,n}$$

and

$$\begin{aligned} \mathcal{C}_2^{m,n} &= \left\{ \mu : \|W_\mu^k(c_2^{m,n}) - c_2^{m,n}\|_1 < \frac{1}{2} \text{ for all } k \geq 0 \right\} \subset \overline{\left\{ \mu : \|\mu - c_2^{m,n}\|_1 < \frac{1}{2} \right\}} \\ &= \overline{\left\{ \mu : \|\mu - (-c_2^{m,n})\|_1 < \frac{1}{2} \right\}} = \overline{-U_2^{m,n}} = A_2^{m,n}. \end{aligned}$$

We show first that if μ is in the complement of $A_i^{m,n}$, then the critical value lies outside of $U_i^{m,n}$. However, this is trivial since the critical value of W_μ is simply μ .

Next, we parameterize the boundary of $A_i^{m,n}$, by $\alpha_i^{m,n}(t)$ where $0 \leq t \leq 2\pi$. We have that $\alpha_1^{m,n}(t)$ goes once around $c_1^{m,n}$ (and $\alpha_2^{m,n}(t)$ goes once around $-c_2^{m,n}$), so $\delta_1^{m,n}(t) = \alpha_1^{m,n}(t) - c_1^{m,n}$ (respectively $\delta_2^{m,n}(t) = \alpha_2^{m,n}(t) - (-c_2^{m,n})$) goes once around zero. In both cases, $i = 1, 2$, $\delta_i^{m,n}$ traces out a diamond about the origin. Applying Theorem 3.3.4 we are done. \square

In Figure 3.7 we show illustrations of $\mathcal{C}_1^{1,0}$ and $\mathcal{C}_1^{2,1}$ respectively.

3.7. Contained Cycles

In this section we discuss the implications of the quadratic-like theory presented earlier. We focus on non-repelling cycles of W_μ contained in $U_i^{m,n}$ for some $i = 1, 2$ and $m, n \in \mathbb{Z}$.

DEFINITION 3.7.1. *We say that the family \wp_λ parameterized over M_{quad} has a contained cycle of type (m, n, i) with multiplier α where $(m, n, i) \in \mathbb{Z}^2 \times \{1, 2\}$ if the map \wp_λ has a cycle $\{p_1, p_2, \dots, p_n\} \subset \phi_\lambda(U_i^{m,n})$.*

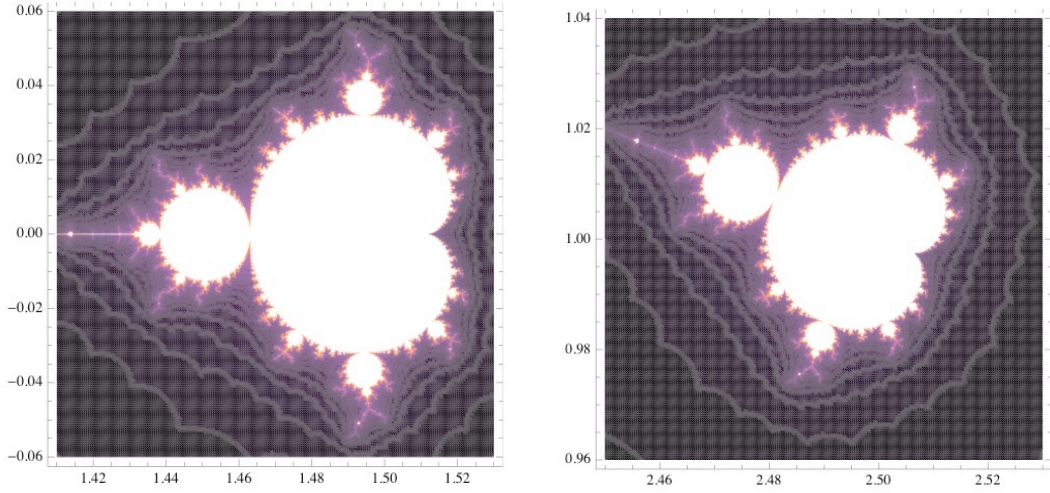


FIGURE 3.7. $\mathcal{C}_1^{1,0}$ and $\mathcal{C}_1^{2,1}$

It is a bit cumbersome to use \wp_λ , so we use W_μ . This is justified by the following statement which is an immediate consequence of the unique conjugacy between \wp_λ and W_μ where $\mu = h(\lambda)$ (n.b. Proposition 3.2.2).

PROPOSITION 3.7.2. *For $\lambda \in M_{quad}$, \wp_λ has a contained cycle $\{p_1, p_2, \dots, p_n\}$ in $\phi_\lambda(U_i^{m,n})$ if and only if $W_\mu = W_{h(\lambda)}$ has a contained cycle $\phi_\lambda^{-1}\{p_1, p_2, \dots, p_n\}$ in $U_i^{m,n}$. Furthermore, these cycles have the same multiplier.*

We can state an equivalent definition:

DEFINITION 3.7.3. *We say that the family \wp_λ parameterized over M_{quad} has a contained cycle of type (m, n, i) with multiplier α where $(m, n, i) \in \mathbb{Z}^2 \times \{1, 2\}$ if the map W_μ has a cycle $\{p_1, p_2, \dots, p_n\} \subset U_i^{m,n}$ with multiplier α .*

It is the case that all super-attracting fixed points are contained; the only thing to check is that if a critical point is fixed, then it is in $U_i^{m,n}$. Since all critical points which can be fixed are $c_1^{m,n}$ or $c_2^{m,n}$, every super-attracting fixed point is of this form. (The

other critical points, $1/2 + i/2 + m + ni$, are pre-poles and so cannot be fixed.) However, it is not the case that all super-attracting cycles are contained. We present the following example to illustrate this.

EXAMPLE 3.7.4. We show that there exist super-attracting period 2 cycles that are not contained. For each m, n we know that there is exactly one which is contained. This is a consequence of Theorem 3.2.19. To find others, which are necessarily not contained, we show that there are solutions to

$$(3.19) \quad W_\mu^2(c_1^{m,n}) = W_\mu(\mu) = c_1^{m,n}$$

with $\mu \notin U_1^{m,n}$. Using the definition of W_μ we can restate Equation 3.19 as

$$(3.20) \quad \frac{\mu}{\gamma^2} \wp_1(\mu) = \frac{1}{2} + m + ni.$$

For simplicity we handle the cases where $n = 0$, $m \in \mathbb{Z}$ and we consider solutions along the positive real axis. Let $f(x) = W_x(x)$ on \mathbb{R} and note that $f(1/2 + m) = 1/2 + m$ for all $m \in \mathbb{Z}$. Furthermore, $f'(1/2 + m) = 1$. Thus, the line $y = x$ is tangent to the graph of $f(x)$ at the points $(1/2 + m, 1/2 + m)$ for each $m \in \mathbb{Z}$. Consider $c_1^{1,0} = 3/2$. We apply the Intermediate Value Theorem applied to the interval $[0, 1/2]$ to give a real value $\mu \in U_1^{0,0}$ so that $f(\mu) = 3/2$. To illustrate the argument we refer the reader to Figure 3.8. We know that $f(1/2) = 1/2 < 3/2$ and $\lim_{x \rightarrow 1^-} f(x) = \infty > 3/2$. Thus, on $[1/2, 1 - \epsilon]$, for some sufficiently small number ϵ , the Intermediate Value Theorem gives a solution to $f(x) = 3/2$ in $[1/2, 1 - \epsilon]$.

In summary, we have a μ so that $3/2$ is a super-attracting cycle of period 2 ($3/2 \rightarrow \mu \rightarrow 3/2$), and $\mu \notin U_1^{1,0}$ with $3/2 \in U_1^{1,0}$.

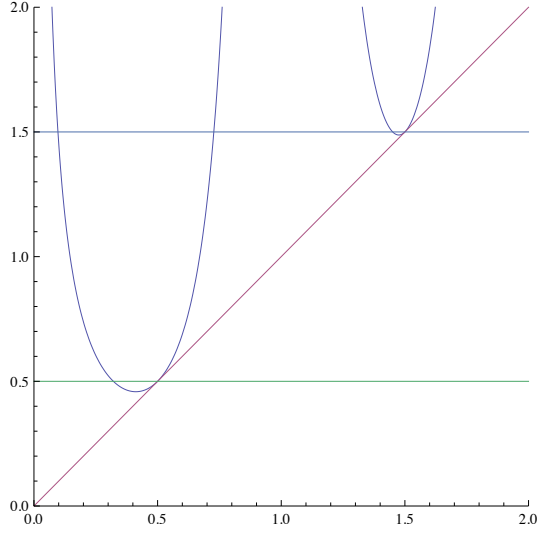


FIGURE 3.8. The graphs of $f(x) = W_x(x)$, $y = x$, and $y = 1/2, 3/2$

There is, in fact, a clustering of solutions to $W_\mu(\mu) = 1/2 + m$ ($m > 0$) on all poles of W_μ . We do not prove this here. Instead, we refer the reader to [34].

The first main result of this section is the following.

THEOREM 3.7.5. *Suppose that W_μ for $\mu \in N_{quad}$ has a contained cycle $\{p_1, p_2, \dots, p_k\}$ of type (m, n, i) with multiplier α . If*

- (1) $|\alpha| < 1$, i.e., $\{p_1, p_2, \dots, p_k\}$ is an attracting cycle,
- (2) $\alpha = e^{2\pi i\theta}$ for $\theta \in \mathbb{Q}$, i.e., $\{p_1, p_2, \dots, p_k\}$ is a parabolic cycle, or
- (3) $\alpha = e^{2\pi i\theta}$ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and W_μ^k is locally linearizable at p_j , i.e., $\{p_1, p_2, \dots, p_k\}$ is a cycle of Siegel points,

then $\mu \in \mathcal{C}_i^{m,n}$.

PROOF. In each case the filled Julia set of the quadratic like map $W_\mu : U_i^{m,n} \rightarrow V_{\pm\mu}$ (or a conformally conjugate map $W_{i^k\mu}$ if μ is not in \hat{N}) is connected [41]. By definition, $\mu \in \mathcal{C}_i^{m,n}$. □

The second main result is the following. As before, let $p_c(z) = z^2 + c$.

THEOREM 3.7.6. *Fix $(m, n, i) \in \mathbb{Z}^2 \times \{1, 2\}$.*

- (1) *For each $c \in \mathcal{M}$ so that p_c has an attracting cycle with multiplier α , there exists a unique $\mu \in \mathcal{C}_i^{m,n}$ so that*

$$W_\mu : K_i^{m,n}(\mu) \rightarrow K_i^{m,n}(\mu)$$

is conformally conjugate to $p_c : K_{p_c} \rightarrow K_{p_c}$, and W_μ has a contained cycle with multiplier α .

- (2) *For each $c \in \mathcal{M}$ so that p_c has a parabolic cycle with multiplier α , there exists a unique $\mu \in \mathcal{C}_i^{m,n}$ so that*

$$W_\mu : K_i^{m,n}(\mu) \rightarrow K_i^{m,n}(\mu)$$

is conformally conjugate to $p_c : K_{p_c} \rightarrow K_{p_c}$, and W_μ has a contained cycle with multiplier α .

- (3) *For each $c \in \mathcal{M}$ so that p_c has a cycle of Siegel points with multiplier α , there exists a unique $\mu \in \mathcal{C}_i^{m,n}$ so that*

$$W_\mu : K_i^{m,n}(\mu) \rightarrow K_i^{m,n}(\mu)$$

is conformally conjugate to $p_c : K_{p_c} \rightarrow K_{p_c}$, and W_μ has a contained cycle of Siegel points with multiplier α .

PROOF. This is immediate from the Straightening Theorem 3.3.3 and the fact that

$$K_i^{m,n}(\mu) \subset U_i^{m,n}.$$

□

The following corollary answers Question 1.7.5.

COROLLARY 3.7.7. *There exists a parameter $\lambda \in M_{quad}$ so that the Fatou set $F(\wp_\lambda)$ of \wp_λ consists of periodic and pre-periodic Siegel disks associated to a contained irrationally neutral cycle.*

PROOF. By Proposition 1.7.7 and Theorem 3.7.6 we have the result. \square

Misiurewicz Points.

The next set of results concern the case when the critical value is strictly pre-periodic. The parameters associated to this phenomenon are traditionally called *Misiurewicz points*. It is necessary that the critical orbit must land on a repelling cycle ([4, Theorem 9.4.4]; see also [30] and [31] for the elliptic case). We discuss the existence of parameters λ so the \wp_λ has a critical value which is strictly pre-periodic and the orbit is contained in $\phi_\lambda(U_i^{m,n})$.

THEOREM 3.7.8. *Suppose that all critical values for \wp_λ are strictly pre-periodic and pre-poles. Then \wp_λ has no non-repelling cycles and thus $J(\wp_\lambda) = \mathbb{C}_\infty$.*

Let l denote Lebesgue measure. Define $\omega(z)$, called the ω -limit set of z , to be

$$\omega(z) = \bigcap_{n=0}^{\infty} \overline{\{f^k(z) : k \geq n\}}.$$

Furthermore, let $S(f)$ denote the set of critical and asymptotic values of f and let

$$P(f) = \bigcup_{n=0}^{\infty} f^n(S(f)).$$

Our motivation for studying Misiurewicz points is the following theorem proved in [37].

THEOREM 3.7.9. [37, Theorem 1] *Let $f : \mathbb{C} \rightarrow \mathbb{C}_\infty$ be a transcendental meromorphic function satisfying the following two conditions:*

$$(1) J(f) = \mathbb{C}_\infty \text{ and}$$

$$(2) l(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0.$$

Then there exists a σ -finite ergodic conservative f -invariant measure equivalent to Lebesgue measure.

If the critical values of \wp_Ω are all pre-periodic and pre-poles then the set described in item 2 is countable. If the critical values that are not pre-poles are strictly pre-periodic then the Julia set is the whole sphere.

In the case of $z^2 + c$, infinity is always a super attracting fixed point, and thus infinity can never be strictly pre-periodic. However, the other critical value c can. In this case the Fatou set is the basin of attraction of infinity and $K_{p_c} = J(p_c)$ [41]. Two examples are $c = i$ and $c = -2$:

$$c = i : i \mapsto -1 + i \mapsto -i \mapsto -1 + i;$$

$$c = -2 : -2 \mapsto 2 \mapsto 2.$$

There is a natural notation for these points which identifies the pre-period and the period.

DEFINITION 3.7.10. We say that a point $c \in \mathcal{M}$ is a Misiurewicz point of type $M_{k,l}$ if

$$p_c^k(c) = p_c^{k+l}(c),$$

where k and l are minimal. Similarly, a point $\mu \in \mathcal{C}_1^{m,n}$ is a Misiurewicz point of type $M_{k,l}$ if

$$W_\mu^k(\mu) = W_\mu^{k+l}(\mu),$$

where k and l are minimal. A point $\mu \in \mathcal{C}_2^{m,n}$ is a Misiurewicz point of type $M_{k,l}$ if

$$W_\mu^k(-\mu) = W_\mu^{k+l}(-\mu),$$

where k and l are minimal.

THEOREM 3.7.11. *Suppose that c is a Misiurewicz point of type $M_{k,l}$ for p_c . Then for each $(m, n, i) \in \mathbb{Z}^2 \times \{1, 2\}$ there is a parameter $\mu \in \mathcal{C}_i^{m,n}$ so that the critical value of W_μ is strictly pre-periodic and μ is a Misiurewicz point of type $M_{k,l}$. Furthermore, the orbit of the critical value is contained in $U_i^{m,n}$.*

PROOF. Applying the Straightening Theorem 3.3.3 to the families

$$W_\mu : U_1^{m,n} \rightarrow V_\mu$$

and

$$\wp_\lambda : U_2^{m,n} \rightarrow V_{-\mu},$$

and keeping in mind the relaxation of the restrictions of (m, n, i) we have the result. \square

COROLLARY 3.7.12. *Suppose that $\mu \in \mathcal{C}_i^{m,n}$ is a Misiurewicz point of type $M_{k,l}$ for W_μ . Letting $\lambda = h^{-1}(\mu)$ for some branch of h^{-1} we have that the critical value of \wp_λ is strictly pre-periodic and the critical value's orbit is contained in $\phi_\lambda(U_i^{m,n})$.*

Disconnected Filled Julia Sets.

We take a more global look at the filled Julia sets. We fix $\lambda \in \hat{M}$ and consider $\phi_\lambda(K_i^{m,n}(h(\lambda)))$ for each $(m, n, i) \in \mathbb{Z}^2 \times \{1, 2\}$. It is known that the Julia set of $p_c(z) = z^2 + c$ is connected if and only if the filled Julia set is connected [41]. This leads to the following question.

QUESTION 3.7.13. *We know that the Julia set $J(\wp_\lambda)$ is always connected. What does this say about the filled Julia sets*

$$\{\phi_\lambda(K_i^{m,n}(h(\lambda)))\}_{(m,n,i) \in \mathbb{Z}^2 \times \{1,2\}}$$

for a fixed λ ?

THEOREM 3.7.14. *For all choices of $(m,n,i) \in \mathbb{Z}^2 \times \{1,2\}$, the filled Julia sets $\phi_\lambda(K_i^{m,n}(h(\lambda)))$ are disconnected, except at most one.*

PROOF. Each filled Julia set $K_i^{m,n}(\mu)$ for W_μ , and thus each $\phi_\lambda(K_i^{m,n}(h(\lambda)))$ for \wp_λ , is pairwise disjoint from any other. Furthermore, the two free critical values, μ and $-\mu$ for W_μ and γ^2/λ^2 and $-\gamma^2/\lambda^2$ for \wp_λ , both map to the same value by evenness. The critical orbit can be contained in at most one filled Julia set. In other words, the tail of the critical orbit, $\{W_\mu^n(\mu)\}_{n \geq 1} = \{W_\mu^n(-\mu)\}_{n \geq 1}$ (similarly for \wp_λ), can be completely contained in at most one $U_i^{m,n}$. Thus, at most one filled Julia set is connected, by Proposition 3.2.20. \square

DEFINITION 3.7.15. *For $\lambda \in \hat{M}$, we say that \wp_λ has a connected filled Julia set if $\phi_\lambda(K_i^{m,n}(h(\lambda)))$ is connected for some $(m,n,i) \in \mathbb{Z}^2 \times \{1,2\}$.*

REMARK 6. *Fix $\lambda \in \hat{M}$ so that $J(\wp_\lambda) = \mathbb{C}_\infty$. Then a \wp_λ may or may not have a connected filled Julia set.*

If the critical values are pre-poles, then the Julia set is the whole sphere, since \wp_λ cannot have any non-repelling cycles. In this case, all filled Julia sets are disconnected. If the critical value is strictly pre-periodic, i.e. λ is a Misiurewicz point, and the orbit of

the critical value is contained, then it is in a filled Julia set and that filled Julia set is thus connected. These parameters exist by Corollary 3.7.12.

REMARK 7. *Fix $\lambda \in \hat{M}$ so that $J(\wp_\lambda) \neq \mathbb{C}_\infty$. Then a \wp_λ may or may not have a connected filled Julia set.*

If there is an attracting cycle which is not contained, then all filled Julia sets $\phi_\lambda(K_i^{m,n}(h(\lambda)))$ are disconnected. This occurs in Example 3.7.4. However, as discussed throughout the chapter, there exist attracting, parabolic, and Siegel cycles which are contained. In fact, for a given $U_i^{m,n}$, these types of cycles (of W_μ) are in one-to-one correspondence with those of $z^2 + c$, by Theorem 3.7.6. In all of these cases \wp_λ has a connected filled Julia set.

CHAPTER 4

Rational Approximations of the Weierstrass Elliptic \wp Function

In the previous chapter we have shown explicit dynamical connections between Weierstrass elliptic functions on square lattices and the classical quadratic map $z^2 + c$. In this chapter we begin exploring connections between higher degree rational maps and Weierstrass elliptic functions on square lattices. The main result of this chapter is Theorem 4.1.5.

PROPOSITION 4.0.1. [1] *For the $\{g_2, g_3\}$ invariant parameterization $\wp_{\{g_2, g_3\}}$, of the Weierstrass elliptic functions we have the following Laurent series centered at zero:*

$$(4.1) \quad \wp_{\{g_2, g_3\}}(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} c_k z^{2k-2},$$

where $c_2 = g_2/20$, $c_3 = g_3/28$, and

$$c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}$$

for $k > 3$. This series converges on $0 < |z| < R$, where $R = \min_{\omega \in \Omega^*} |\omega|$.

Our family of interest is the similarity class of square lattices. Thus, we have that $g_3 = 0$ and $g_2 = 4\gamma^4/\lambda^4$, where λ generates the lattice $\langle \lambda, \lambda i \rangle$. This gives the following.

PROPOSITION 4.0.2. [1] *For \wp_λ we have the following Laurent series centered at zero:*

$$(4.2) \quad \wp_\lambda(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k} z^{4k-2} = \frac{1}{z^2} + \frac{\gamma^4}{5\lambda^4} z^2 + \cdots,$$

where $c_2 = \gamma^4 5\lambda^4$ and

$$c_{2k} = \frac{3}{(4k+1)(2k-3)} \sum_{m=1}^{k-1} c_{2m} c_{2(k-m)}$$

for $k > 1$. This series converges on $0 < |z| < |\lambda|$.

By truncating the infinite Laurent series we obtain parameterized families of rational approximations to \wp_λ . The following families of rational maps of degree $n+d$ are studied in [6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein:

$$(4.3) \quad D_{\alpha,n,d}(z) = z^n + \frac{\alpha}{z^d}.$$

We show that these families are conformally conjugate to the following families:

$$(4.4) \quad P_{\beta,n,d}(z) = \beta z^n + \frac{1}{z^d}$$

where $\beta = \alpha^{\frac{n-1}{d+1}}$. These maps are the first two terms of the Laurent series of an elliptic function of order d .

PROPOSITION 4.0.3. *Let $\phi_{\alpha,d}(z) = \alpha^{1/(d+1)}z$, where $\alpha^{1/(d+1)}$ is any branch. Then*

$$\phi_{\alpha,d}^{-1} \circ D_{\alpha,n,d} \circ \phi_{\alpha,d} = P_{\alpha^{\frac{n-1}{d+1}},n,d}.$$

PROOF. We show the direct calculation:

$$\begin{aligned} (\phi_{\alpha,d}^{-1} \circ D_{\alpha,n,d} \circ \phi_{\alpha,d})(z) &= \frac{1}{\alpha^{1/(d+1)}} \left((\alpha^{1/(d+1)}z)^n + \frac{\alpha}{(\alpha^{1/(d+1)}z)^d} \right) \\ &= \alpha^{(n-1)/(d+1)} z^n + \frac{1}{z^d} = P_{\alpha^{\frac{n-1}{d+1}},n,d}(z). \end{aligned}$$

□

There are symmetries in both the dynamical and parameter spaces of $P_{\beta,n,d}$. We begin with the parameter space.

PROPOSITION 4.0.4. *Let $a_p = e^{2\pi i/p}$. Then $P_{\beta,n,d}$ is conformally conjugate to $P_{a_{d+1}^{n-1}\beta,n,d}$. Thus, the parameter space is symmetric with respect to the multiplicative group generated by a_{d+1}^{n-1} .*

PROOF. Let $\phi_{e^{2\pi i},d}(z) = a_{d+1}z$ as in 4.0.3. Then

$$\left(\phi_{e^{2\pi i},d}^{-1} \circ P_{\beta,n,d} \circ \phi_{e^{2\pi i},d}\right)(z) = \frac{1}{a_{d+1}} \left(\frac{1}{(a_{d+1}z)^d + \beta(a_{d+1}z)^n} \right) = P_{a_{d+1}^{n-1}\beta,n,d}.$$

□

This is a general result which we will apply to the maps $P_{\beta,n,d}$.

THEOREM 4.0.5. *Let $f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be a rational map. Suppose that for each z , $f(\alpha z) = \alpha^k f(z)$ for some integer k and some $\alpha \in \mathbb{C}$ with $\alpha^m = 1$ for some m . If z lies in the basin of attraction of some attracting cycle, then αz also lies in the basin of attraction of an attracting cycle.*

PROOF. Let $\alpha \in \mathbb{C}$ with $\alpha^m = 1$ for some m and fix z so that $f^{nd}(z) \rightarrow w$ as $n \rightarrow \infty$, where w is an attracting fixed point of f^d . We begin with two claims.

- (1) For $l > 0$ we have $f(\alpha^l z) = \alpha^{lk} f(z)$, and
- (2) for $n > 0$ $f^n(\alpha z) = \alpha^{k^n} f(z)$.

The first claim follows by induction on l . The $l = 1$ case is our assumption. Observe that $f(\alpha^l) = \alpha^k f(\alpha^{l-1})$. The induction hypothesis gives $\alpha^k f(\alpha^{l-1}) = \alpha^k \alpha^{k(l-1)} = \alpha^{lk} f(z)$, which proves the first claim. The second claim uses induction on n . Again, the $n = 1$

case is our assumption. Assume that $f^{n-1}(\alpha z) = \alpha^{k^{n-1}} f^{n-1}(z)$. Our first claim gives us that $f(\alpha^{k^{n-1}} f^{n-1}(z)) = \alpha^{k^{n-1}k} f^n(z) = \alpha^{k^n} f^n(z)$ which gives the second claim.

We compute that $f^{nd}(\alpha z) = \alpha^{k^{nd}} f^{nd}(z)$, giving $|f^{nd}(\alpha z) - \alpha^{k^{nd}} w| \rightarrow 0$ as $n \rightarrow \infty$.

The set $\{\alpha^{k^{nd}} w | n \in \mathbb{N}\}$ is

- (1) finite,
- (2) forward f^d invariant, and
- (3) attracting.

The finiteness follows from the fact that $\{\alpha^n | n \in \mathbb{Z}\}$ forms a finite group of order $\leq m$. The forward invariance follows from the calculation $f^d(\alpha^{k^{nd}} w) = \alpha^{k^{(n+1)d}} f(w) = \alpha^{k^{(n+1)d}} w$. Consider $|\{\alpha^{k^{nd}} w | n \in \mathbb{N}\}| = p$. Now $f^{dp}(\alpha^{k^{nd}} w) = \alpha^{k^{nd}} w$. We need to show that $|(f^{dp})'(\alpha^{k^{nd}} w)| < 1$. We have that

$$\frac{d}{dz} f^{dp}(\alpha^{k^{nd}} z) = \frac{d}{dz} \alpha^{k^{dp} k^{nd}} f^{dp}(z) = \alpha^{k^{(n+p)d}} \frac{d}{dz} f^{dp}(z).$$

Taking the modulus and evaluating at $z = w$, we have

$$|(f^{dp})'(\alpha^{k^{nd}} w)| = \prod_{i=1}^p |(f^d)'(f^{id}(w))| = |(f^d)'(w)|^p < 1,$$

so the cycle is attracting. □

COROLLARY 4.0.6. *If f satisfies the hypotheses of 4.0.5 and $\mathcal{F}(f)$ is the union of the basins of attracting cycles, then $\alpha^k \mathcal{F}(f) = \mathcal{F}(f)$ and $\alpha^k \mathcal{J}(f) = \mathcal{J}(f)$ for all $k \in \mathbb{Z}$.*

As a result we have the following theorem.

THEOREM 4.0.7. *For $a_p = e^{2\pi i/p}$, we have that $a_{n+d}^k \mathcal{F}(P_{\beta,n,d}) = \mathcal{F}(P_{\beta,n,d})$ and $a_{n+d}^k \mathcal{J}(P_{\beta,n,d}) = \mathcal{J}(P_{\beta,n,d})$ for all $k \in \mathbb{Z}$*

PROOF. We check that $P_{\beta,n,d}(a_{n+d}z) = a_{n+d}^{-d}P_{\beta,n,d}(z)$ and, using Corollary 4.0.6, we have our result. \square

4.1. Connectivity of the Julia Set

Motivated by the connectivity results of Chapter 2, we prove a similar set of results for the family of maps $P_{\beta,2,2}$. The $n = 2 = d$ case corresponds to the first two terms in Laurent series for \wp_λ . To maintain consistency we define the following family:

$$(4.5) \quad P_\lambda(z) = P_{\gamma^4/(5\lambda^4),2,2}(z) = \frac{1}{z^2} + \frac{\gamma^4}{5\lambda^4}z^2,$$

since

$$(4.6) \quad \wp_\lambda(z) = P_\lambda(z) + O\left(\frac{z^6}{\lambda^8}\right)$$

from Proposition 4.0.2.

Just as with \wp_λ , P_λ satisfies a homogeneity equation:

$$(4.7) \quad P_\lambda(\lambda z) = \frac{1}{\lambda^2}P_1(z).$$

This family also has similar mapping properties.

DEFINITION 4.1.1. *Fix $\lambda \in \mathbb{C}^*$. Define*

$$L_1 = \{t\lambda^{-2} : t \in \mathbb{R}^+\}$$

and

$$L_2 = -L_1.$$

Furthermore, define

$$\tilde{S}_1 = \{t\lambda : t \in \mathbb{R} \setminus \{0\}\}.$$

Set

$$\tilde{S}_2 = i\tilde{S}_1.$$

PROPOSITION 4.1.2. *For $\lambda \in \mathbb{C}^*$ we have that*

$$P_\lambda(\tilde{S}_1) \subset L_1$$

and

$$P_\lambda(\tilde{S}_2) \subset L_2.$$

PROOF. If $z \in \tilde{S}_1$, then $z = t\lambda$ for some $t \in \mathbb{R} \setminus \{0\}$. By the homogeneity equations (4.7), we have that

$$P_\lambda(t\lambda) = \lambda^{-2}P_1(t).$$

Furthermore, $P_1(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, giving that $\lambda^{-2}P_1(t) \in L_1$.

If $z \in S_2$, then $z = itk$ for $t \in \mathbb{R} \setminus \{0\}$. We have that $P_{i\lambda} = P_\lambda$. Using the homogeneity equations a second time, we have

$$P_\lambda(it\lambda) = P_{i\lambda}(it\lambda) = (i\lambda)^{-2}P_1(t) = -\lambda^{-2}P_1(t).$$

Since $P_1(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, we have that $-\lambda^{-2}P_1(t) \in L_2$. \square

The critical points of P_λ are $\{z : z^4 = \frac{5\lambda^4}{\gamma^4}\}$ and the point at infinity. The finite critical points can be written as $i^k \frac{\sqrt[4]{5\lambda}}{\gamma}$, where $k = 0, 1, 2$, or 3 . The point at infinity is a super-attracting fixed point. This leaves one free critical orbit, since after two iterations all critical points (except infinity) land on the same point. Thus P_λ can have at most one finite non-repelling periodic cycle [4].

THEOREM 4.1.3. *Suppose that P_λ has a finite attracting or parabolic cycle. Then each Fatou component contains at most one critical value.*

PROOF. This proof proceeds identically to the proof of Theorem 2.3.6. \square

The following theorem appears without proof in [16].

THEOREM 4.1.4. [16] *If the critical values of P_λ do not tend to infinity then $J(P_\lambda)$ is connected.*

We present the following theorem.

THEOREM 4.1.5. *Suppose that P_λ has an attracting cycle, parabolic cycle, cycle of Siegel disks, or a strictly pre-periodic critical value. Then $J(P_\lambda)$ is connected.*

PROOF. If P_λ has an attracting cycle or parabolic cycle then Theorem 2.3.6 gives that each Fatou component contains at most one critical value. If P_λ has a cycle of Siegel disks or a strictly pre-periodic critical value then the critical values are contained in the Julia set, and so again, each Fatou component contains at most one critical value. By applying [40, Theorem 3.1] we have the result. (We recall that Theorem 2.1.2 is an adaptation of [40, Theorem 3.1].) \square

4.2. The Parameter Space of P_λ

Just as in the setting with \wp_λ Chapter 3, we introduce a new map that simplifies the description of the critical points and critical values.

We show that the following family of maps is conjugate to the family P_λ where $\mu = \frac{\gamma^3}{5^{3/4}\lambda^3}$:

$$(4.8) \quad R_\mu(z) = \frac{\mu}{8} \left(\frac{1}{z^2} + 16z^2 \right).$$

A brief calculation shows the following result.

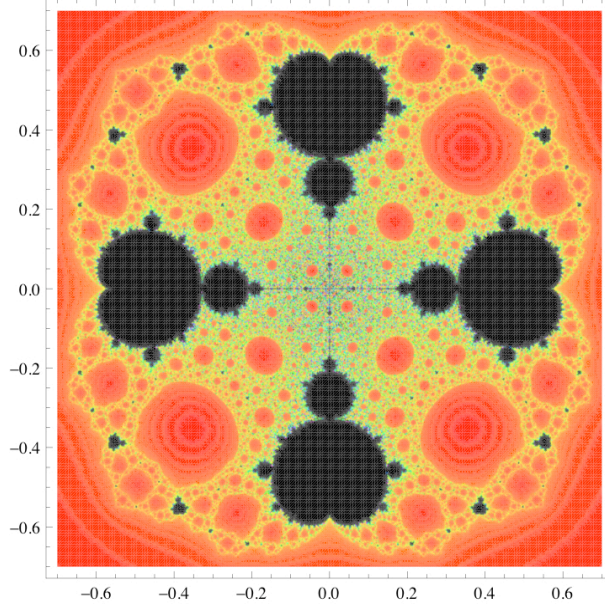


FIGURE 4.1. The parameter plane of R_μ

PROPOSITION 4.2.1. *The finite critical points of R_μ are $\{1/2, i/2, -1/2, -i/2\}$, and the critical values are $\{-\mu, \mu\}$.*

Let

$$\tilde{h}(z) = \frac{\gamma^3}{5^{3/4} z^3}$$

and set $\psi_\lambda(z) = \frac{2\lambda}{5^{1/4}\gamma} z$. Using the above maps we have the following.

PROPOSITION 4.2.2. *The map $R_{\tilde{h}(\lambda)}$ is conformally conjugate to P_λ via ψ_λ .*

PROOF. We show that $R_{\tilde{h}(\lambda)} = \psi_\lambda^{-1} \circ P_\lambda \circ \psi_\lambda$:

$$\begin{aligned} (\psi_\lambda^{-1} \circ P_\lambda \circ \psi_\lambda)(z) &= \frac{1}{\frac{2\lambda}{5^{1/4}\gamma}} \left(\frac{1}{\left(\frac{2\lambda}{5^{1/4}\gamma} z\right)^2} + \frac{\gamma^4}{5\lambda^4} \left(\frac{2\lambda}{5^{1/4}\gamma} z\right)^2 \right) \\ \frac{\tilde{h}(\lambda)}{8z^2} + 2\tilde{h}(\lambda)z^2 &= \frac{\tilde{h}(\lambda)}{8} \left(\frac{1}{z^2} + 16z^2 \right). \end{aligned}$$

□

It is known that for $n > 2$, the parameter space of $D_{\lambda,n,n}$ contains $n - 1$ copies of the Mandelbrot set along the rays $\{te^{2k\pi i/(n-1)} : t > 0, 0 \leq k < n - 1\}$ [12]. However, the quadratic-like mapping theory fails in the case of $D_{\lambda,2,2}$. It is conjectured by Devaney in [12] that there is one Mandelbrot set in the parameter plane centered on the super attracting parameter with the “tip” missing. This tip corresponds to the parameter $c = -2$. We present a picture of the parameter plane of R_μ in Figure 4.2. If R_μ has a bounded critical orbit, then the point μ is colored black. This family is conformally conjugate to $D_{\lambda,2,2}$. Since we have several maps under consideration, we present the conjugacy explicitly here.

PROPOSITION 4.2.3. *The map $D_{\mu^4,2,2}$ is conformally conjugate to R_μ via the map $\phi_\mu(z) = z/(2\mu)$.*

PROOF. We show $\phi_\mu^{-1} \circ R_\mu \circ \phi_\mu = D_{\mu^4,2,2}$:

$$(\phi_\mu^{-1} \circ R_\mu \circ \phi_\mu)(z) = 2\mu \frac{\mu}{8} \left(\frac{4\mu^2}{z^2} + \frac{1}{4\mu^2} z^2 \right) = D_{\mu^4,2,2}(z).$$

□

CHAPTER 5

Future Work

In this chapter we describe work in progress and describe unanswered questions relevant to the work in this thesis.

The Connectedness of $J(\wp_\Omega)$.

In this section we assume that $\Omega = \langle \omega_1, \omega_2 \rangle$ is any lattice, not necessarily square. We have established in Theorem 2.2.1 that $J(\wp_\Omega)$ is connected when Ω is square. It has been shown by Hawkins and Koss in [33, 32] that $J(\wp_\Omega)$ is connected when Ω is triangular. We have the following conjecture of Hawkins, Koss, et al.:

CONJECTURE 5.0.1. *Let Ω be any lattice. Then $J(\wp_\Omega)$ is connected.*

If \wp_Ω satisfies the hypotheses of Theorem 2.1.2 then $J(\wp_\Omega)$ is connected. This naturally leads one to ask if Theorem 2.1.2 is a strong enough sufficient condition. It is shown in [33] that there are lattices so that a Fatou component contains two critical values. These produce so called “toral band” Fatou components. It is not known if the complementary Julia sets are connected. Also, It is not known whether there is a lattice so that three critical values of \wp_Ω lie in the same Fatou component. However, if this occurs, the Julia set is necessarily disconnected [33].

Buried Mandelbrot Sets.

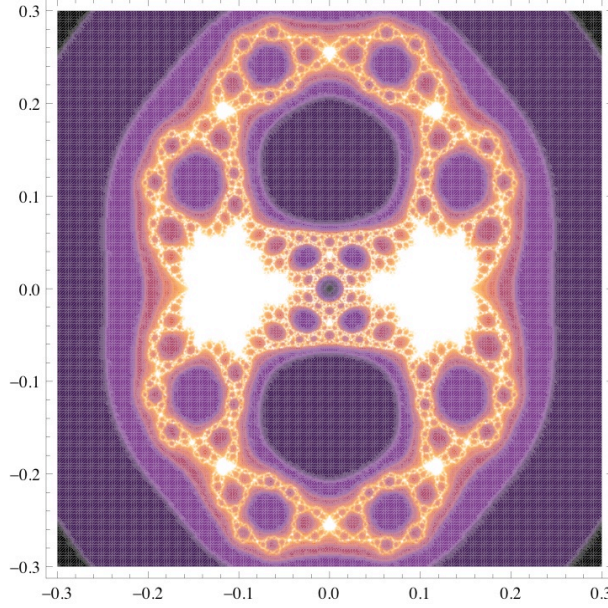


FIGURE 5.1. The parameter plane of $D_{\lambda,3,3}$

As a model for the discussion of the parameter space of W_μ we will first discuss the parameter space of

$$D_{\lambda,n,n}(z) = z^n + \frac{\lambda}{z^n}.$$

It is known that for $n > 2$, the parameter space of $D_{\lambda,n,n}$ contains $n - 1$ copies of the Mandelbrot set along the rays $\{te^{2k\pi i/(n-1)} : t > 0, 0 \leq k < n - 1\}$ [12]. It is conjectured that there are smaller “buried” Mandelbrot sets in the parameter space. These sets are disjoint from the $n - 1$ “baby” Mandelbrot sets. One can clearly see smaller “Mandelbrot sets” in the parameter plane of $D_{\lambda,3,3}$ in Figure 5 in addition to the 2 baby Mandelbrot sets. Additionally, one can see these conjectured Mandelbrot sets in the parameter plane of R_μ in Figure 4.2.

We conjecture that W_μ (and so \wp_λ) also has these buried Mandelbrot sets in the parameter space. If Figure 5 we color the parameter μ white if the critical orbit remains bounded in the diamonds $\|z\|_1 < 1, 2, 3$ respectively.

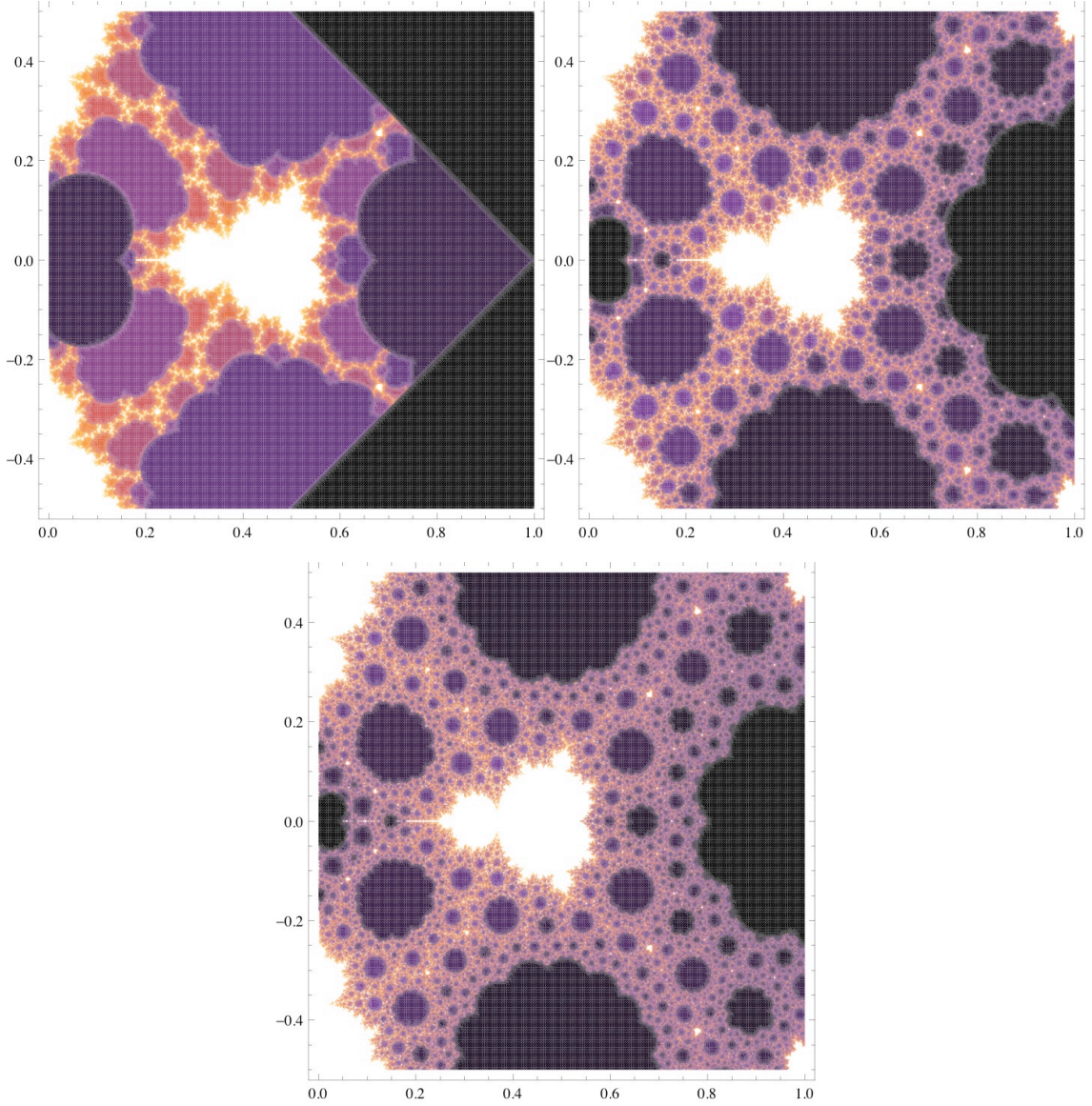


FIGURE 5.2. μ so that $\|W_\mu^n(\mu)\|_1 < 1, 2, 3$ respectively for $n < 200$.

We have established that there is an embedding of the Mandelbrot set into the parameter plane of W_μ (and so also \wp_λ) which respects the dynamics. We conclude with the following question inspired by Figure 5:

QUESTION 5.0.2. *Is there an embedding (or partial embedding) of the parameter space of R_μ into the parameter plane of W_μ which respects the dynamics?*

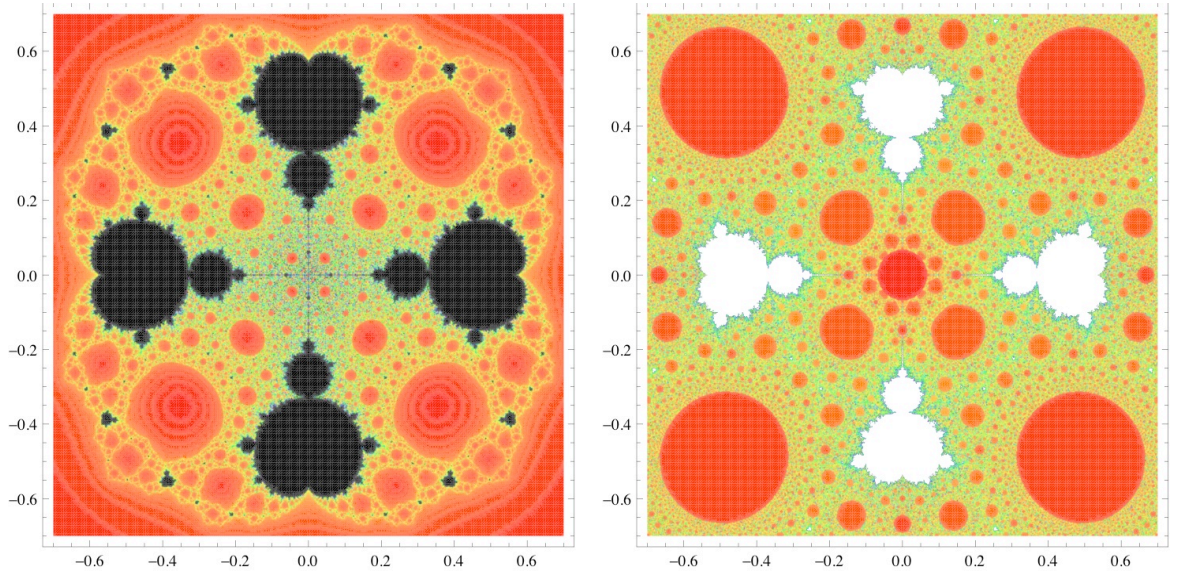


FIGURE 5.3. (left) The parameter plane of R_μ indicating when the critical orbit is bounded
 (right) The parameter plane of W_μ indicating when the orbit of the critical value is bounded in a large disk of radius 10

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